INTRODUCTORY NOTES ON VECTOR GEOMETRY

By K. White

Illustrated by Susie Boyd

PREFACE

It has become increasingly apparent that elementary notes on Vector Geometry which utilize vector methods for solutions of problems in Plane, Solid and Analytic geometry whilst at the same time introducing some of the concepts of Matrices and Linear algebra gently to the tyro seem to be sadly lacking.

One usually finds that the modern textbooks immediately start with the definition of a linear vector space, linear dependence, independence and the like and generalize these concepts almost immediately. Actually, the student usually hasn't the vaguest idea whatever as to why he or she are proving these theorems except forpurposes of passing an examination or obtaining another credit.

Books are available on Matrices and Determinants, on Analytic Geometry, on Calculus with vectors, on vector geometry, etc., but they all seem to have something missing. It is the author's purposeto combine these concepts with the concept of a vector and to use vector methods to solve many varieties of problems taken mostly from geometry. Many examples are given at the end of each section, but once the students familiarize themselves with a few vector basics they will find that a little ingenuity will solve otherwise extremely difficult problems in geometry.

The author does not dwell on proofs of well-known theorems. Indeed, there are a plethora of texts where the proofs of these theorems may be found. Instead, the important theorems will be stressed and the accent on the <u>use</u> of these theorems will be emphasized.

Also it must be accentuated that this text panders to the intuitive approach and those that are seeking or striving for a rigorous approach had better eke out another text. On the other hand, the author does not wish to be accused of being slipshod but wishes to propose that a little intuition and a few well-chosen examples do far more for the beginner than a host of beautiful generalizations and techniques which the beginner can neither appreciate because of lack of sophistication in the discipline nor

apply to a particular example in a particular situation, which more often than not occurs in a job situation and believe me, I know from experience that the employer is not reimbursing the average employee to entertain him with elegant mathematical histrionics!

Some of the more profound theorems are dealt with in a rather cursory and simple-minded fashion, the chief emphasis being on solving practical geometry problems. This was deliberate and the reasons are twofold: (1) this is an introductory course and is taken at the Junior College level (CEGEP in Quebec) and it is the author's intention that although some of these topics are dealt with in a cavalier manner that there should be enough curiousity aroused in the better student to pique his(or her) interest and to engender further mental activity in some other particular areas. (2) the reader might note some bias towards a geometrical approach in general and the author must confess that this is also deliberate since he feels(and hopes) that ultimately physical descriptions will once again be most adequately described by geometry rather than by probability theory.

In any case it is sincerely hoped that this effort will satisfy the need for an introductory course whilst at the same time provide the necessary foundation for subsequent excursions(if any) into Differential Geometry, Engineering Mathematics, Tensor Analysis or any other related topic.

The usual procedure with an average class was to cover all of chapters 1-12 inclusive and if the class showed minimum enthusiasm to forge ahead with chapters 13 and 14. A good basic calculus course should be prerequisite but by suitable editing, the course might conceivably be given without this. Frankly, though, I would not want to take an abridged version of the course if I were a student since it detracts from the concatenation of ideas as herewith presented. It was found that the order as given in these notes was the most expedient and is recommended as a procedure in giving the course over one semester although I feel sure that there will be some who read these lines that will wish to present the material differently to suit their style of teaching.

My special thanks to Miss Susie Boyd whose invaluable assistance aided immeasurably in clarifying the concepts by means of the elegant illustrations presented in these pages.

K. White

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CHAPTER 1

REVIEW OF PLANE ANALYTIC GEOMETRY

1. THE GENERAL EQUATION OF THE SECOND DEGREE IN 2-SPACE(2 DIMENSIONS)

The general equation of the second degree in 2 variables(for 2-space) is $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$. This equation represents a conic section i.e. a straight line, circle, parabola, ellipse or hyperbola. First, let us recall the definition of these curves:

(1) Straight Line - A curve which represents the shortest distance between two points, in 2 dimensions or in the plane. Also, a curve whose slope is everywhere the same. Its equation always has to be of the first degree.

Example 1 $x + y = 4$

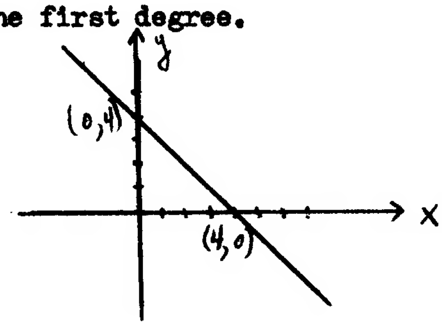


FIGURE 1

To make the plotting easy, find the intercepts, i.e. let x and y be zero alternately and find out where they cross the x and y-axes respectively. For the above example when $y = 0$, $x = 4$ and when $x = 0$, $y = 4$. Since two points determine a straight line the curve is quickly drawn.

(2) CIRCLE - A curve which is determined by a set of points always equidistant from some fixed point called the center. The fixed distance is known as the radius and the curve is shown in Figure 2 below.

(3) PARABOLA - A curve which denotes the set of points which are equidistant from a fixed point(called the focus) and a fixed line(called the directrix). This curve is illustrated in figure 3 below.

(4) ELLIPSE - A curve which denotes a set of points where the sum of the distances from two fixed points(called focuses or foci) is always equal to a constant value. This curve is illustrated in figure 4 below.

(5) HYPERBOLA - A curve which denotes a set of points such that the difference of the distances from two fixed points(called the foci) is always the same. This curve is illustrated in figure 5 below.

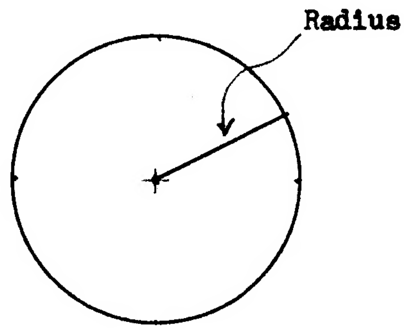
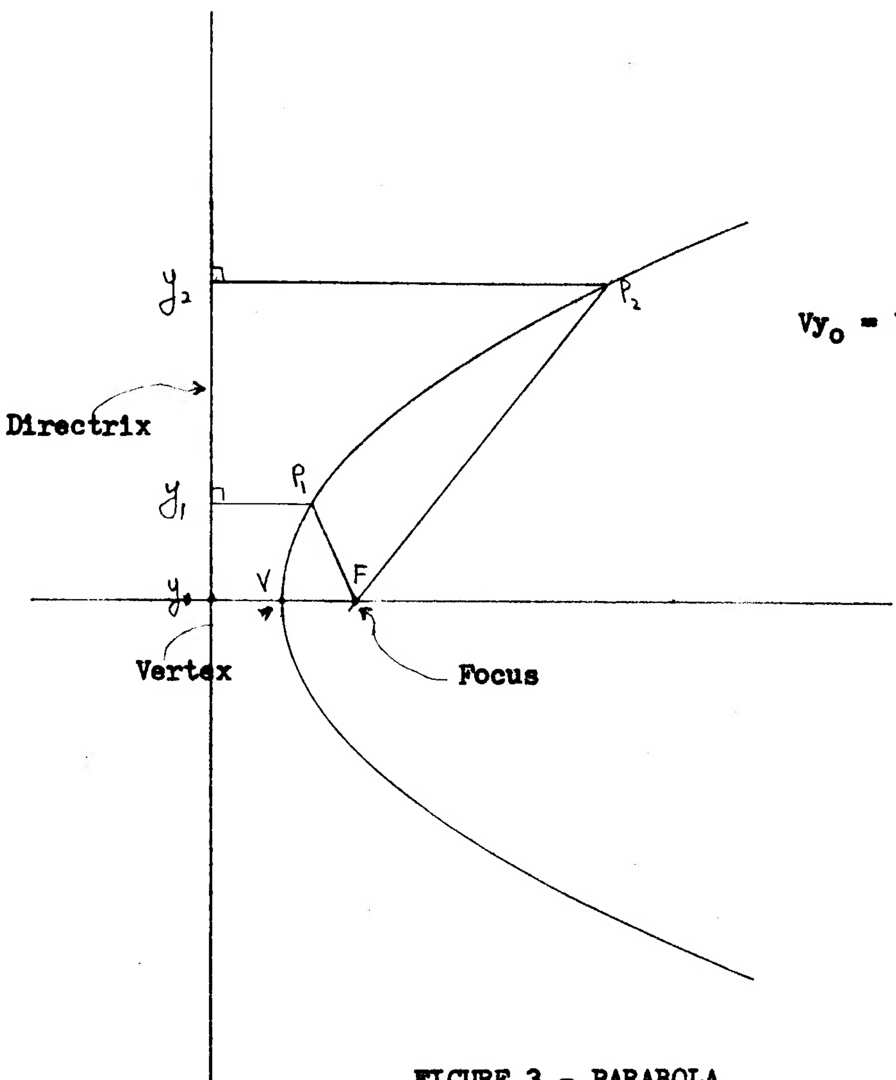
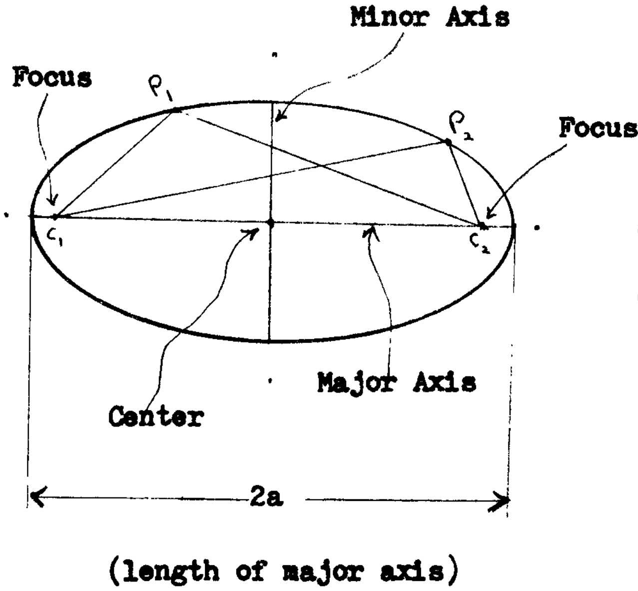


FIGURE 2 - CIRCLE



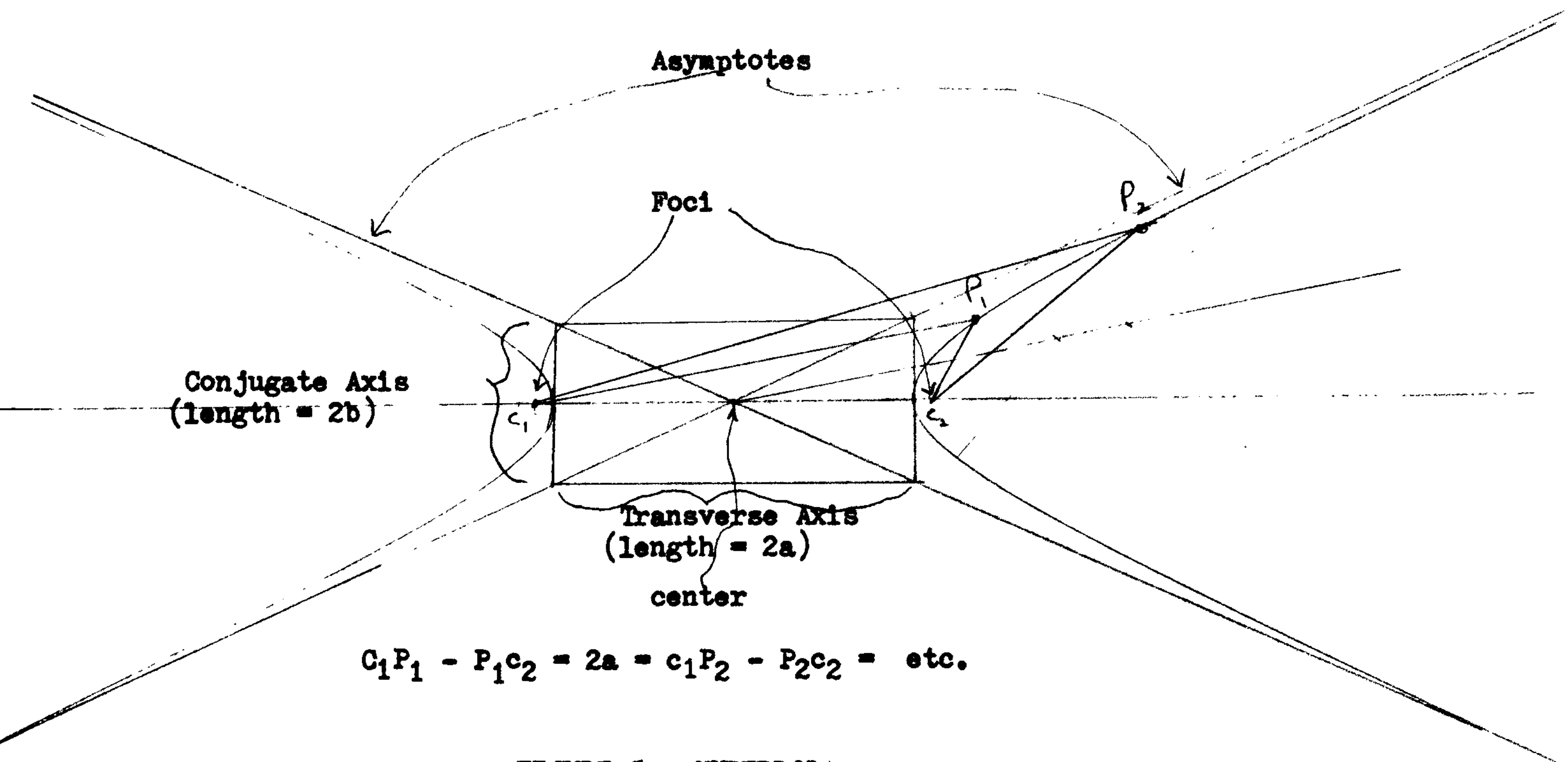
$Vy_0 = VF$, $P_1y_1 = P_1F$, $P_2y_2 = P_2F$, etc.

FIGURE 3 - PARABOLA



$$c_1P_1 + P_1c_2 = 2a$$
$$c_1P_2 + P_2c_2 = 2a \quad \text{etc.}$$

FIGURE 4 - ELLIPSE



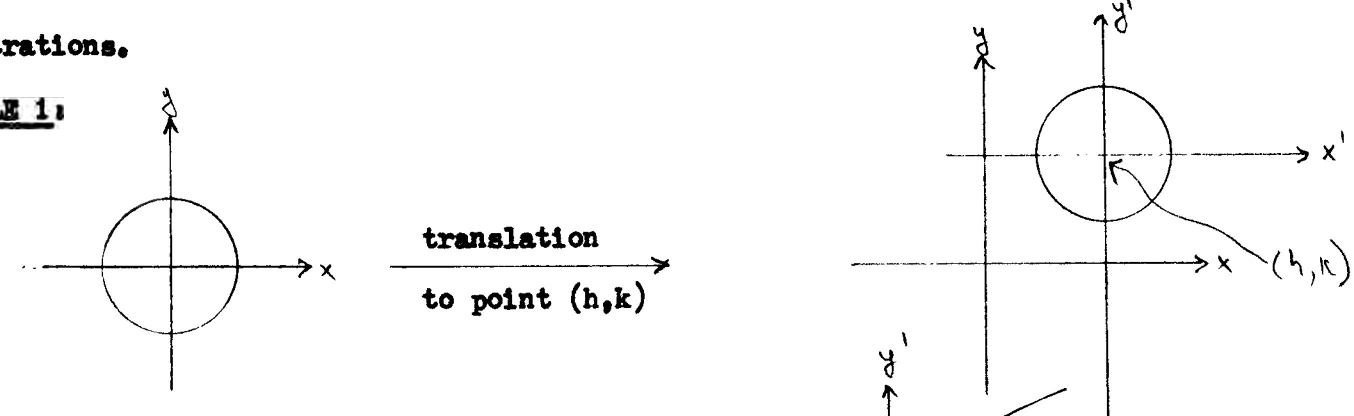
$$C_1P_1 - P_1c_2 = 2a = c_1P_2 - P_2c_2 = \text{etc.}$$

FIGURE 5 - HYPERBOLA

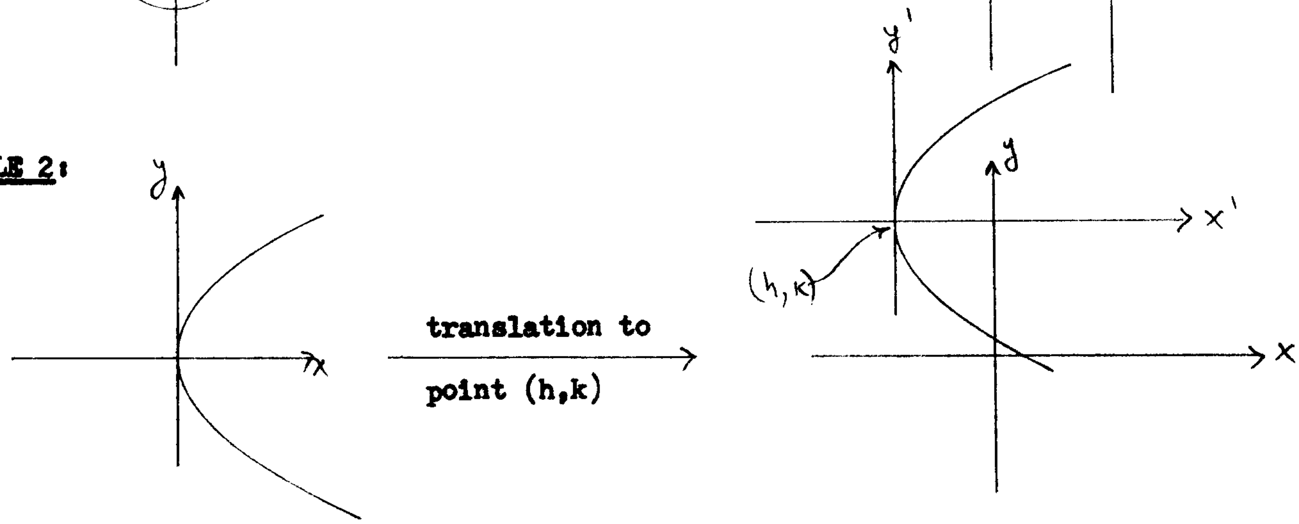
2. IDENTIFICATION OF CONICS IN 2-SPACE

There are two sets of rules for identification of conic sections. These rules only apply to equations of the second degree! First, if an equation of the second degree has an xy term, it must involve rotation. Conversely, if there is no xy term present, there is no rotation, but there might be translation involved. Again, let us recall what we mean by translation and rotation of axes by means of some pictorial illustrations.

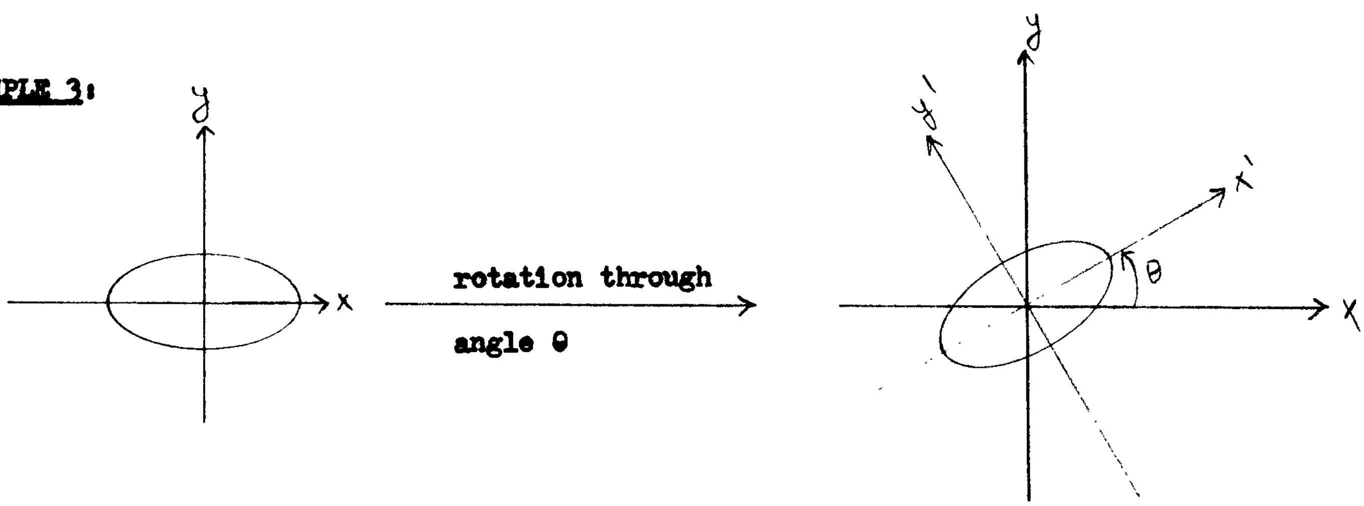
EXAMPLE 1:



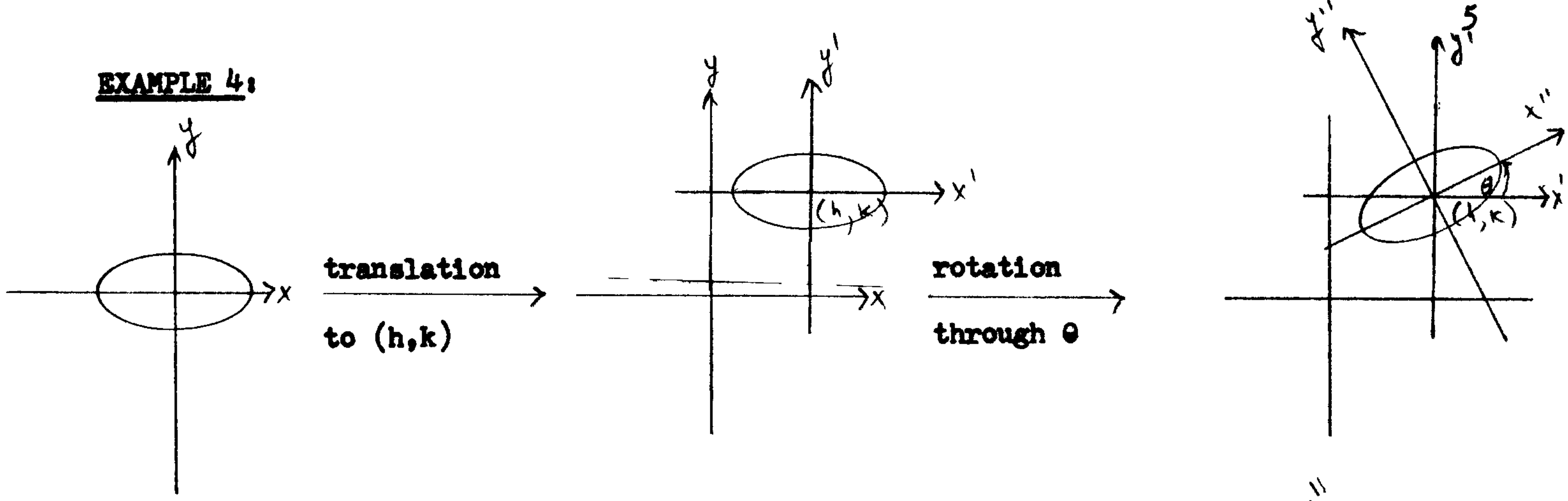
EXAMPLE 2:



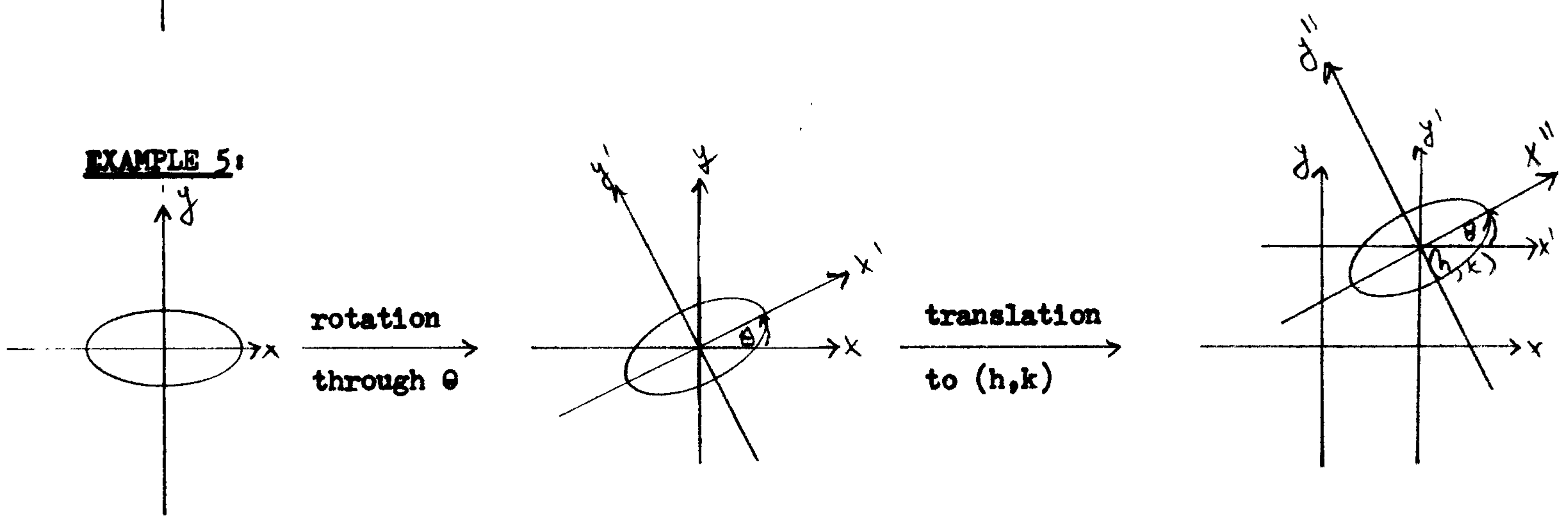
EXAMPLE 3:



EXAMPLE 4:



EXAMPLE 5:



It is seen that it makes no difference whether we translate and then rotate or rotate and then translate. The set of rules for use with equations involving rotation i.e. equations containing an xy term in $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$ are listed below.

(1) If $B^2 - AC < 0$, then the curve is an ellipse.

(2) If $B^2 - AC = 0$, then the curve is a parabola.

(3) If $B^2 - AC > 0$, then the curve is an Hyperbola.

EXAMPLE 6: Identify the curve $2x^2 - y^2 - 3xy + 5x + 3y - 2 = 0$.

Here, $A = 2$, $2B = -3$, $C = -1$. Accordingly, we have,

$B^2 - AC = \left(-\frac{3}{2}\right)^2 - (2)(-1) = \frac{9}{4} + 2 > 0$. Therefore the curve is an hyperbola.

EXAMPLE 7: Identify the curve $x^2 - 2xy + y^2 - 3x + 2y - 5 = 0$.

Here $A = 1$, $2B = -2$, $C = 1$. Accordingly, we have,

$B^2 - AC = (1)^2 - (1)(1) = 0$. Therefore the curve is a parabola.

Note that since these are general rules they do not obviate the possibility of the curve "degenerating" i.e. a curve breaking down into points or simpler curves.

EXAMPLE 8: Identify the curve $x^2 - 3xy + 2y^2 = 0$

Here, $A = 1$, $2B = -3$, $C = 2$ and $B^2 - AC > 0$ which indicates that the curve is an hyperbola but $x^2 - 3xy + 2y^2 = 0$ factors into $(x - 2y)(x - y) = 0$ and hence $x = 2y$ or $x = y$ and thus the curve represents two straight lines.

EXAMPLE 9: Identify the curve $x^2 - 2xy + y^2 - 2x + 2y - 3 = 0$.

Here, $A = 1$, $B = -1$, $C = 1$ and $B^2 - AC = 0$ which indicates that the curve would be a parabola but by rearranging terms we have, $(x-y)^2 - 2(x-y) - 3 = (x-y-3)(x-y+1) = 0$ and this again represents two straight lines.

Thus one must always be on the lookout for degenerate curves i.e. one must be able to determine whether $f(x,y)$ is factorable or not and clearly, in the above cases a little elementary algebra can quickly determine this.

Regarding the set of rules for use with equations containing no xy term i.e. where there is no rotation involved, we have the following:

(1) $|A|$ and $|B| = 0$ indicates a straight line i.e. a linear equation.

(2) $|A| = |B|$ and A, B have same sign, then the curve is a circle.

(3) $|A| \neq |B|$ and A, B have same sign, then the curve is an ellipse.

(4) $|A| = 0$ or $|B| = 0$(but not both), then the curve is a parabola.

(5) $|A| = |B|$ or $A \neq B$ and A, B have different signs, then the curve is an hyperbola.

EXAMPLE 10: Identify the curve $x^2 + 3y^2 - 5x + 6y - 2 = 0$.

Here $|A| = 1$, $|B| = 3$ and A,B have the same sign. Hence the curve is an ellipse by (3)

EXAMPLE 11: Identify the curve $3x^2 + 3y^2 - 5x + 6y - 2 = 0$.

Here $|A| = 3$, $|B| = 3$ and A,B have the same sign, hence the curve is a circle by (2).

EXAMPLE 12: Identify the curve $-3x^2 + 3y^2 - 5x + 6y - 2 = 0$.

Here $|A| = 3$, $|B| = 3$ but A,B have different signs, hence the curve is an hyperbola by (5).

Example 13: Identify the curve $x^2 - 5x + 6y - 2 = 0$.

Here $|A| = 1$, $|B| = 0$ and thus the curve is a parabola by (4).

EXERCISES (1) Identify the following conics:

1. $2x^2 - 7xy + 3y^2 - 33x + 5y - 6 = 0$.
2. $6x^2 + 3xy + 2y^2 - 38 = 0$.
3. $49x^2 + 4y^2 = 196$.
4. $16x^2 - 25y^2 - 16x + 5y - 3 = 0$.
5. $4x^2 - 12xy + 9y^2 - 2y + 3x - 11 = 0$.
6. $xy = 5$
7. $10x - 3y - 3x^2 - 2y^2 + 5 = 0$.
8. $x^2 - y^2 = 0$.
9. $3y^2 - 5y + x - 2 = 0$.
10. $3y^2 + 5y - 12 = 0$.
11. $x^2 - 4xy + 4y^2 - x + 2y - 12 = 0$.
12. $2x^2 - 7xy - 22y^2 - 5x + 35y - 3 = 0$.
13. $5x^2 + 3y^2 - 10x - 12y - 40 = 0$.
14. $x^2 + y^2 = -1$.
15. $x^2 - y^2 = -1$.
16. $xy - x = 0$.
17. $x^2 - 1 = 0$.
18. $x^2 - y^2 = 0$.
19. $x^2 - y = 1$.
20. $x^2 - 2xy - y^2 - 3y + 2x = 0$.

3. FUNDAMENTAL FORMS OF CONICS IN 2-SPACE WITHOUT ROTATION OR TRANSLATION

Essentially there are 4 fundamental forms(not including the straight line). The origin is, of course, considered to be in the most convenient location for ease in plotting. The forms are shown below and derivation of these forms can be obtained easily with or without vector methods but will be postponed until a later chapter where examples will be given utilizing vector methods.

(1) THE CIRCLE: $x^2 + y^2 = r^2$

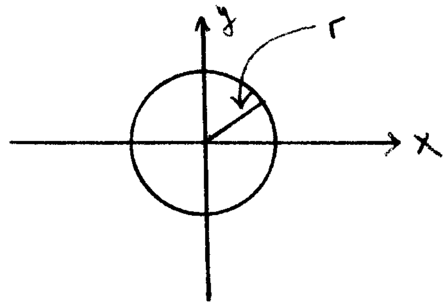
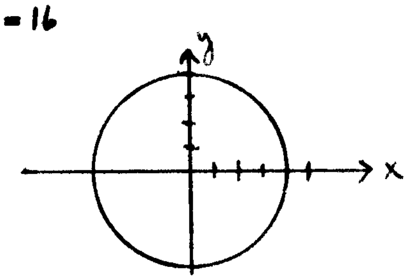


FIGURE 6

EXAMPLE 1: $x^2 + y^2 = 16$



(2) THE ELLIPSE: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

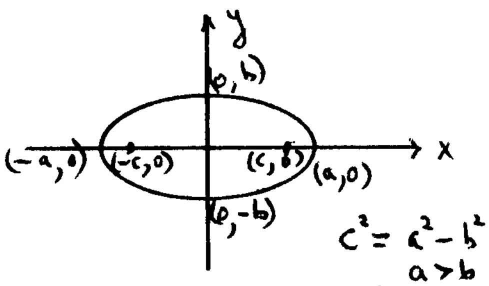


FIGURE 7(case 1)

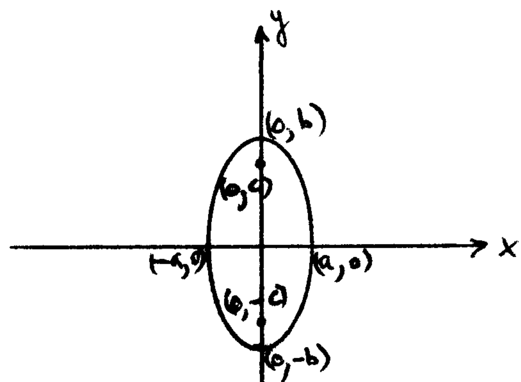


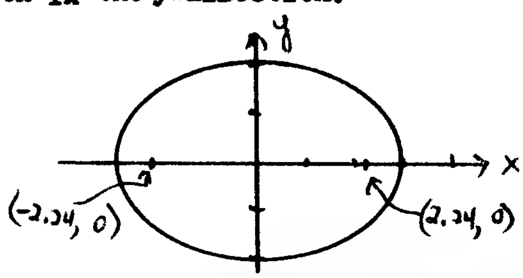
FIGURE 8(case 2)

In some texts the second case(figure 8) takes on the form $x^2/b^2 + y^2/a^2 = 1$. This form is convenient for using formulas to find eccentricity, directrices and so forth, since the formulas will remain the same for both cases. However, for a quick plot, the above method is usually employed and 2a is always considered the length in the x direction, 2b the length in the y direction.

EXAMPLE 2: $x^2/9 + y^2/4 = 1$

$c^2 = 9 - 4 = 5$

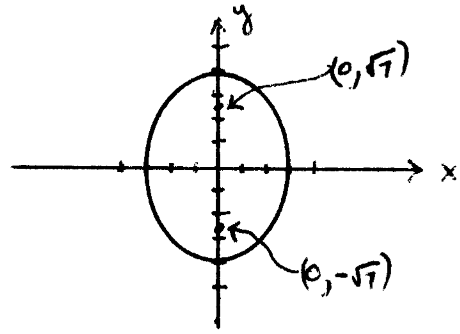
$c = \sqrt{5} \cong 2.24$



EXAMPLE 3: $x^2/9 + y^2/16 = 1$

$c^2 = 16 - 9$

$c = \sqrt{7} = 2.65$



(3) **THE PARABOLA:** There are four cases for the parabola corresponding to whether it opens left, right, up or down. Each case is represented by a different equation.

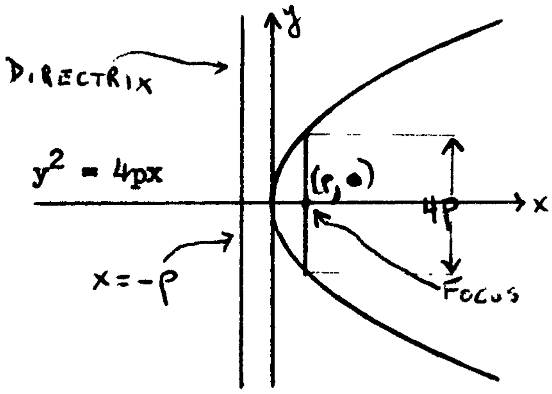


FIGURE 9(case 1)

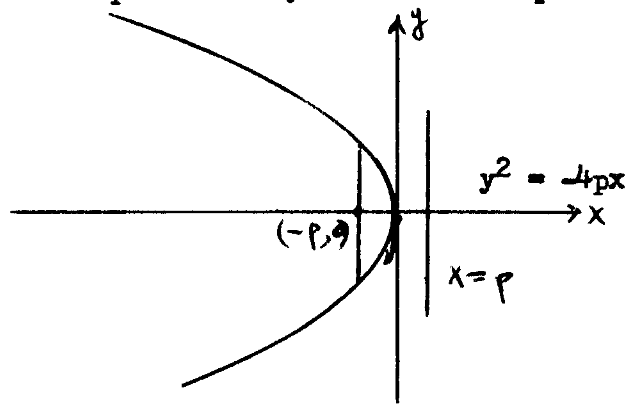


FIGURE 10(case 2)

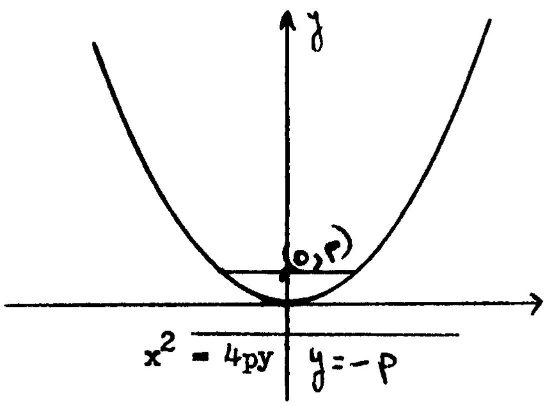


FIGURE 11(case 3)

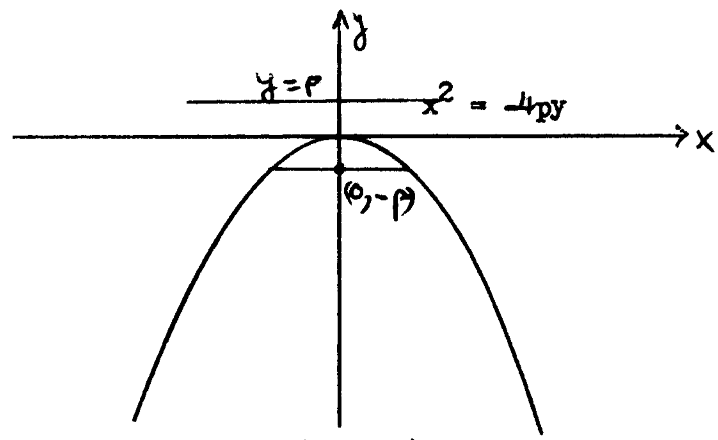
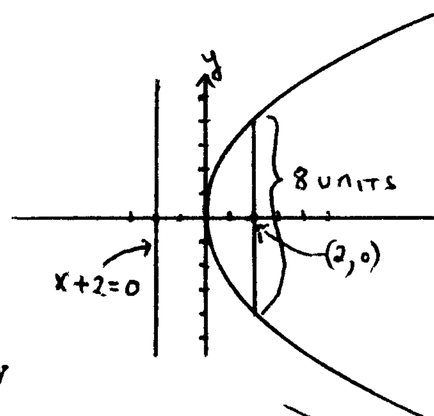


FIGURE 12(case 4)

EXAMPLE 4: $y^2 = 8x$

$4p = 8$

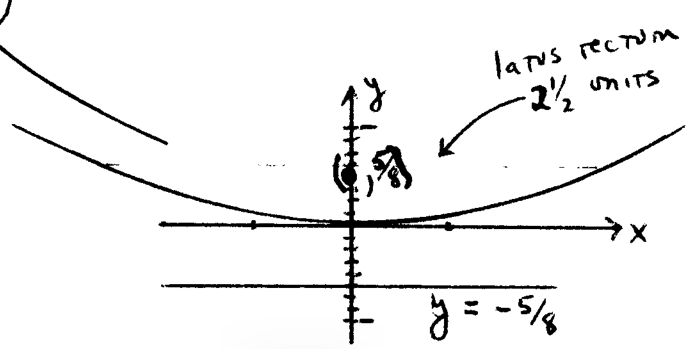
$p = 2$



EXAMPLE 5: $2x^2 = 5y$

$4p = 5/2$

$p = 5/8$



Summarizing, we have: $y^2 = \pm 4px$ (right +ve, left -ve)

$x^2 = \pm 4py$ (up +ve, down -ve)

(4) THE HYPERBOLA:

Just as in the case of the general equation of the ellipse, the hyperbola has two fundamental forms. The first being the form where the hyperbola opens to the right and left and the second being where the hyperbola opens up and down. Both of these forms are shown in figures 13 and 14 below:

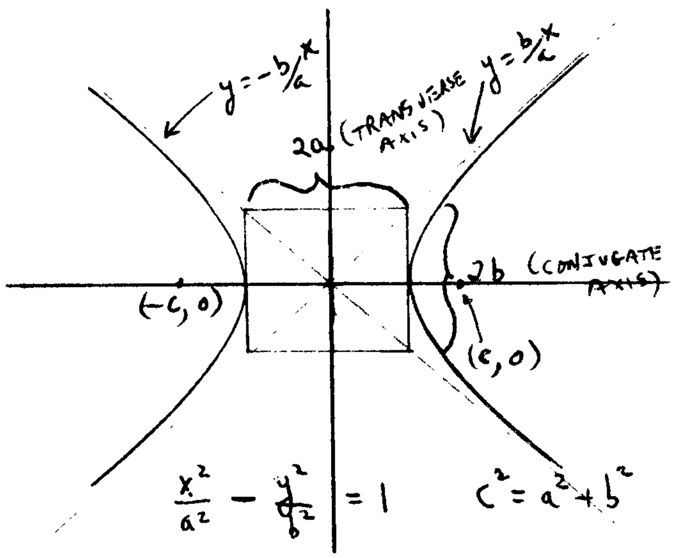


FIGURE 13(case 1)

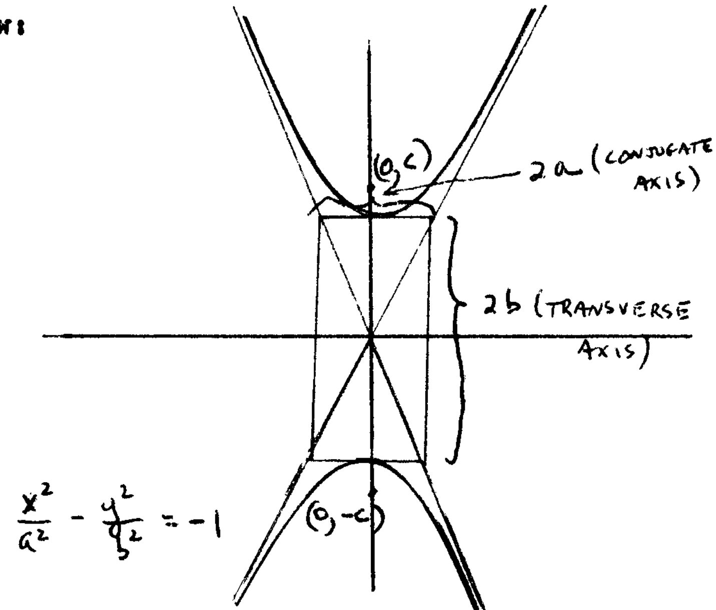
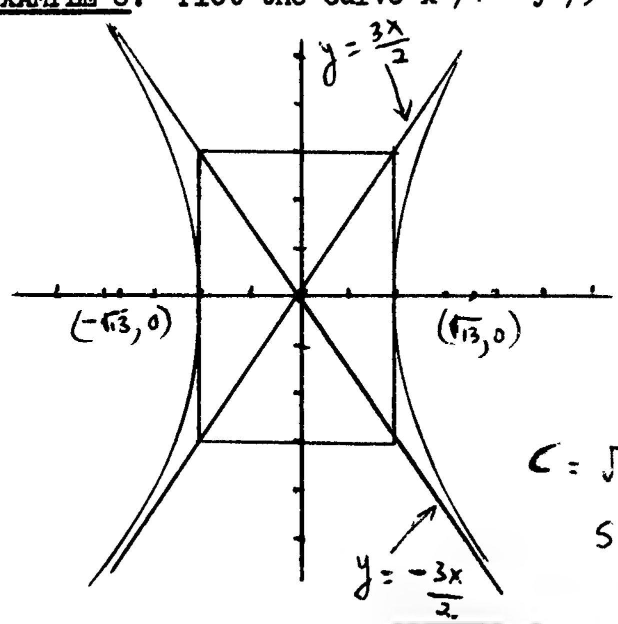


FIGURE 14(case 2)

Recall that asymptotes are "tangents at infinity" to a curve i.e. a straight line that a curve continuously approaches but never touches. One again must be careful when applying formulas for eccentricity and the like since the roles of a and b have been reversed(as in the cases of the ellipse). To find the equations of the asymptotes, simply let the right hand side be zero of the given equation.

EXAMPLE 6: Plot the curve $x^2/4 - y^2/9 = 1$



First, construct a 2 x 3 rectangle.

Second, draw asymptotes(must intersect at origin).

Third, draw curve opening to right and left or up and down according to whether the sign of 1 is +ve or -ve.

Fourth, calculate distances of foci from origin(in this case $c^2 = 13$).

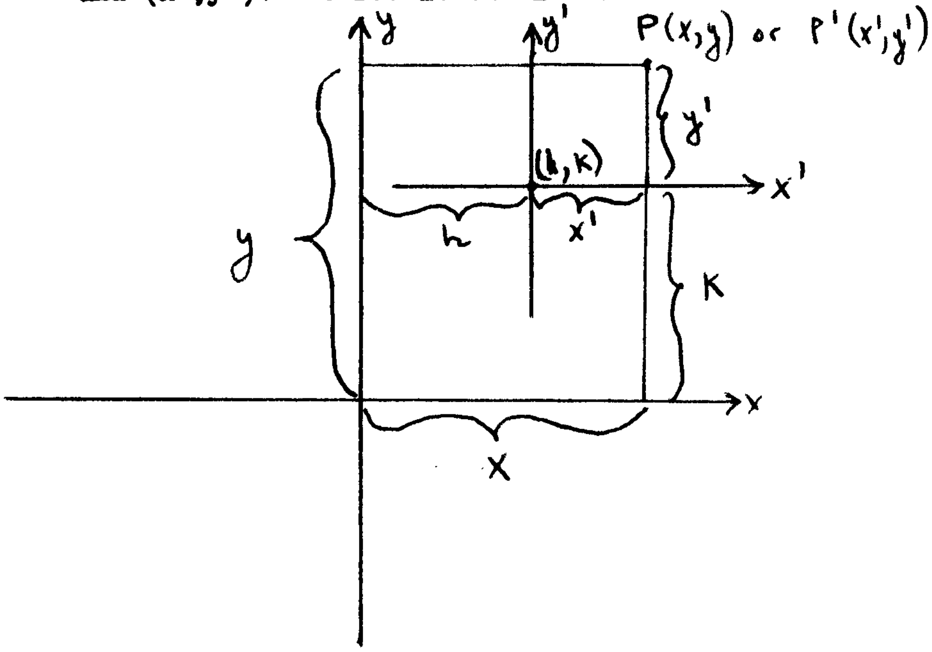
Fifth, find equations of asymptotes.

$c = \sqrt{9+4} = \sqrt{13} \cong 3.6$

Set $\frac{x^2}{4} - \frac{y^2}{9} = 0$ to find equations of asymptotes.

4. DERIVATION OF EQUATIONS OF TRANSLATION IN 2-SPACE

Suppose we wish to translate (or move) the origin to the point (h,k) so that a given point (x,y) with respect to the original coordinate system becomes (x',y') in the new coordinate system. We ask the question, what is the relationship between (x,y) and (x',y')? The result should be clear from FIGURE 15.



From the drawing it is apparent that $X' + h = x$ and $y' + k = y$. Hence, we have,

$$x' = x - h \qquad x = x' + h$$

$$y' = y - k \qquad y = y' + k$$

FIGURE 15

5. FUNDAMENTAL FORMS OF CONICS IN 2-SPACE WITH TRANSLATION

(1) CIRCLE - Recall that $x^2 + y^2 = r^2$ represented a circle whose center was at the origin with radius r. Now suppose that we wished to have the center at some point (h,k) i.e. we wish to "pretend" that the origin is at (h,k) instead of (0,0). This means that our final form would still be simple-in fact it would be $x'^2 + y'^2 = r^2$ which would greatly expedite plotting the curve. To obtain the equation in this form we merely have to use the equations of translation - namely, $x' = x - h$, $y' = y - k$. Hence the form of the circle now becomes $(x-h)^2 + (y - k)^2 = r^2$ with reference to the old coordinate system. Similarly, it can readily be seen that the forms of the ellipse, parabola and hyperbola become:

(2) ELLIPSE - $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

(3) PARABOLA - $\underset{\text{(right - left)}}{(y - k)^2 = \pm 4p(x - h)}$; $\underset{\text{(up - down)}}{(x - h)^2 = \pm 4p(y - k)}$

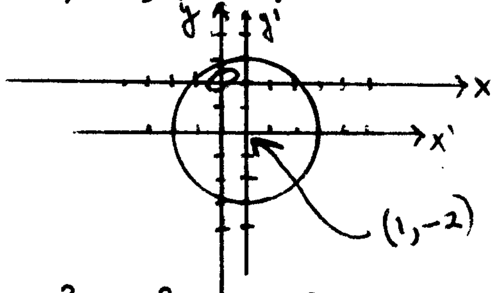
(4) HYPERBOLA - $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = \pm 1$ (− up & down; + right & left)

The only difficulty now is to be able to put the conic equations in the fundamental forms shown above so that plotting becomes elementary. This is done by "completing the square".

<u>EXAMPLE 1</u>: Plot $x^2 + y^2 - 2x + 4y - 4 = 0$.

We have $x^2 - 2x + 1 + y^2 + 4y + 4 = 4 + 1 + 4 = 9$ i.e.

$(x - 1)^2 + (y + 2)^2 = 3^2$. Thus, we have a circle whose center is (1,-2) and radius = 3.

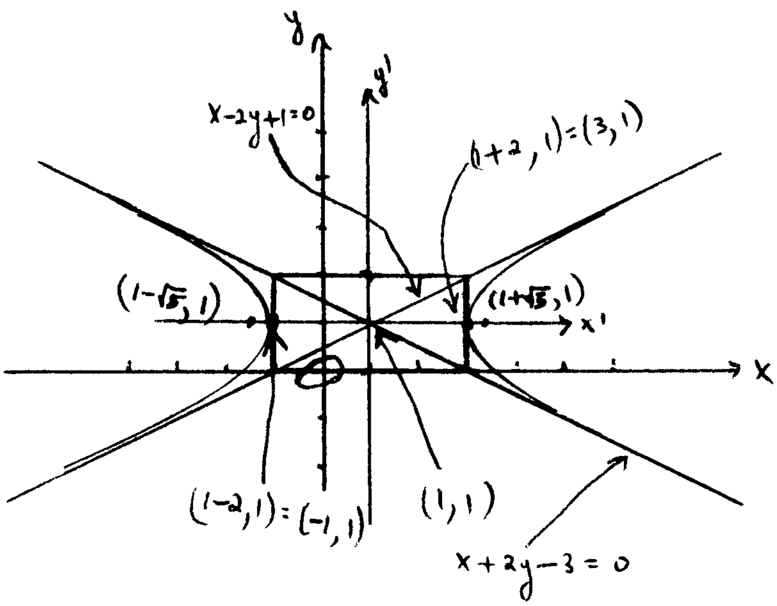


<u>EXAMPLE 2</u>: Plot $x^2 - 4y^2 - 2x + 8y - 7 = 0$.

We recognize this as an hyperbola and completing the square, we obtain,

$x^2 - 2x + 1 - 4(y^2 - 2y + 1) = 7 + 1 - 4 = 4$. This gives $(x - 1)^2 - 4(y - 1)^2 = 4$.

Dividing by 4,, we obtain $\dfrac{(x - 1)^2}{4} - \dfrac{(y - 1)^2}{1} = 1$. Accordingly, we have the center at (1,1) and the rectangle equal to 2 by 1.



$c^2 = 4 + 1 = 2.24^2$.

Equations of asymptotes are

$\dfrac{x - 1}{2} = \pm\dfrac{y - 1}{1}$, Hence,

$x - 2y + 1 = 0$; $x + 2y - 3 = 0$.

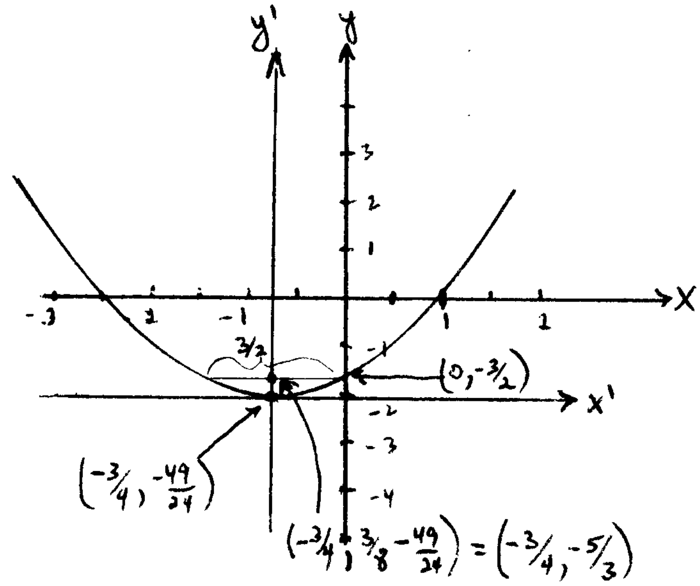
Note that c,b and a values have to be added to the coordinates of the new origin!

<u>EXAMPLE 3</u>: Plot $2x^2 - 3y + 3x - 5 = 0$.

This is a parabola, of course, and we complete the square. We have,

$2(x^2 + 3/2x + 9/16) = 3y + 5 + 9/8$. This gives $(x + 3/4)^2 = 3/2(y + 49/24)$.

Thus the parabola opens up and the vertex is at (-3/4,-49/24). The latus rectum length is 3/2. Also, note that the y-intercept is at (0,-5/3) and the x-intercepts are at (-5/2,0), (1,0) which are obtained by solving the quadratic $2x^2 + 3x - 5 = 0$.



EXERCISES (2): Plot the following conics:

1. $x^2 + y^2 = 3$.
2. $x^2 = 4y$.
3. $4x^2 - 9y^2 = 36$.
4. $2x^2 + 3y^2 = 8$.
5. $2y - x - 2x^2 - 2y^2 + 3 = 0$.
6. $x^2 + y^2 = 0$.
7. $x^2 - y^2 = 0$.
8. $9x^2 + 4y^2 = -36$.
9. $9x^2 - 4y^2 - 36x - 24y - 36 = 0$.
10. $y^2 - 4x - 4y + 4 = 0$.
11. $x^2 + 4y^2 + 2x - 16y + 13 = 0$.
12. $y = x^2 - 2x + 1$.
13. $3y^2 - 2y + 1 = 0$.
14. $9x^2 - 4y^2 - 18x + 16y - 7 = 0$.
15. $x^2 + 3y^2 - 5x + 6y - 20 = 0$.
16. $x^2 - 2x + 3y - 5 = 0$.
17. $3x^2 + 4y^2 - 12x + 8y + 16 = 0$.
18. $16x^2 + 3y^2 - 64x + 6y + 55 = 0$.
19. $3x^2 + 3y^2 - 12x + 4y - 6 = 0$.
20. $9x^2 - 4y^2 + 18x + 16y + 29 = 0$.

6. DERIVATION OF EQUATIONS OF ROTATION IN 2-SPACE

Suppose we wish to move axes through an angle θ in 2-space; what is the relationship between (x',y') and (x,y) where (x',y') is the point with reference to new axes and (x,y) is the point with respect to the original axes. To obtain this relationship we construct a drawing and with the use of some basic trigonometry, we can immediately derive the equations of rotation. This is illustrated in figure 15 below:

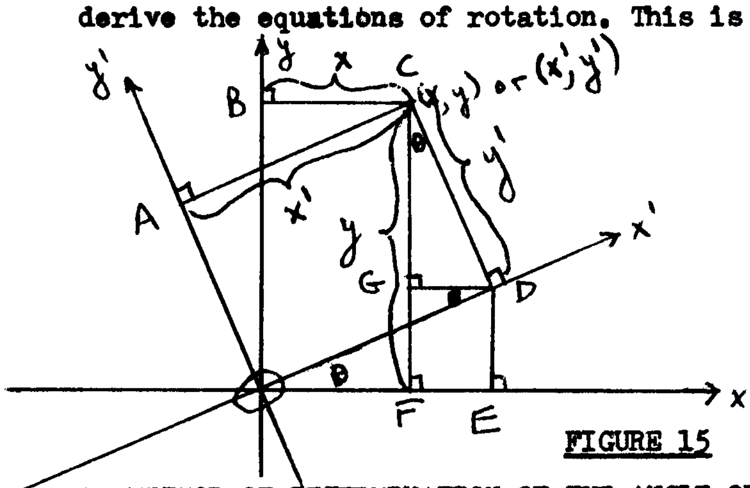


FIGURE 15

$OF = x = OE - FE = OE - GD$

but $OE = OD\cos\theta = x'\cos\theta$

$GD = y'\sin\theta$. Hence $\underline{x = x'\cos\theta - y'\sin\theta}$

$CF = y = CG + GF = CG + DE$

but $CG = y'\cos\theta$, $DE = x'\sin\theta$.

Hence, $\underline{y = x'\sin\theta + y'\cos\theta}$

7. METHOD OF DETERMINATION OF THE ANGLE OF ROTATION SO AS TO OBVIATE THE CROSS-PRODUCT TERM

To determine the angle of rotation we merely substitute the equations of rotation into the general equation of the second degree and set the coefficient of the cross-product term xy = 0. Since only the second degree terms involve any cross-product terms,

we need not worry about the first degree terms, i.e. we will have,

$$Ax^2 + 2Bxy + Cy^2 = A(x'\cos\theta - y'\sin\theta)^2 + 2B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$

$+ C(x'\sin\theta + y'\cos\theta)^2$ and omitting the terms containing x^2 and y^2 we have the xy coefficients as follows:

$(-2A\cos\theta\sin\theta)x'y' + 2B(\cos^2\theta - \sin^2\theta)x'y' + (2C\sin\theta\cos\theta)x'y'$ but since $2\sin\theta\cos\theta = \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and setting the coefficient of the term containing $x'y' = 0$, we have,

$(C - A)\sin 2\theta + 2B\cos 2\theta = 0$ which gives $\boxed{\tan 2\theta = \frac{2B}{A - C}}$. Naturally, we try to choose θ in such a way as to make it an acute angle!

8. REDUCTION OF CONICS WITH CROSS-PRODUCT TERM TO FUNDAMENTAL FORM

To simplify an equation with a cross-product term(of the 2nd degree), we utilize the relationships established in 7. above.

EXAMPLE 1: Simplify $5x^2 + 8xy + 5y^2 - 9 = 0$ by a suitable rotation.

First, we determine that the curve is an ellipse since $B^2 - AC < 0$. Next, we see that $\tan 2\theta = 8/0 = \infty$. Hence, $2\theta = 90^\circ$ and therefore $\theta = 45^\circ$ (taking the least acute angle).

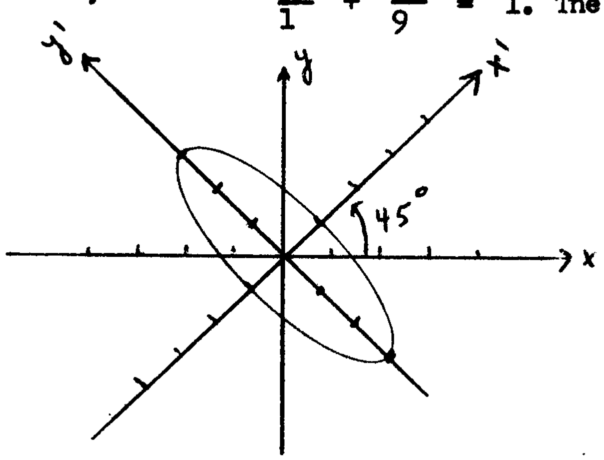
Accordingly, our equations of rotation are:

$x = x'\cos\theta - y'\sin\theta = x'/\sqrt{2} - y'/\sqrt{2}$; $y = x'\sin\theta + y'\cos\theta = x'/\sqrt{2} + y'/\sqrt{2}$.

Substituting in the original equation above and dropping the primes for convenience, we have,

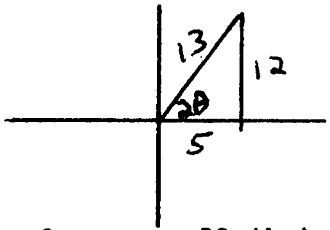
$$\frac{5(x - y)^2}{2} + \frac{8(x - y)(x + y)}{2} + \frac{5(x + y)^2}{2} - 9 = 0.$$ This gives

$5(x^2 - 2xy + y^2) + 8(x^2 - y^2) + 5(x^2 + 2xy + y^2) - 18 = 0$, which results in $18x^2 + 2y^2 - 18 = 0$ or $9x^2 + y^2 - 9 = 0$. Replacing primes and putting the equation in the standard form, we obtain $\frac{x'^2}{1} + \frac{y'^2}{9} = 1$. The graph is shown below:



EXAMPLE 2: Plot the curve $4x^2 + 12xy - y^2 + 80 = 0$.

First, we determine that this equation represents an hyperbola since $B^2 - AC$ is greater than zero. Next, we determine that $\tan 2\theta = 12/5$ and placing 2θ in the 1st quadrant to assure that θ will be an acute angle, we have,



$\cos 2\theta = 5/13$

To determine θ, we recall that $\cos\theta = \sqrt{1 + \cos 2\theta}/\sqrt{2}$ and $\sin\theta = \sqrt{1 - \cos 2\theta}/\sqrt{2}$

$\therefore \cos\theta = \sqrt{\dfrac{1 + 5/13}{2}} = \dfrac{3}{\sqrt{13}}$; $\sin\theta = \sqrt{\dfrac{1 - 5/13}{2}} = \dfrac{2}{\sqrt{13}}$

Accordingly, the equations of rotation are $x = \dfrac{3x'}{\sqrt{13}} - \dfrac{2y'}{\sqrt{13}}$; $y = \dfrac{2x'}{\sqrt{13}} + \dfrac{3y'}{\sqrt{13}}$.

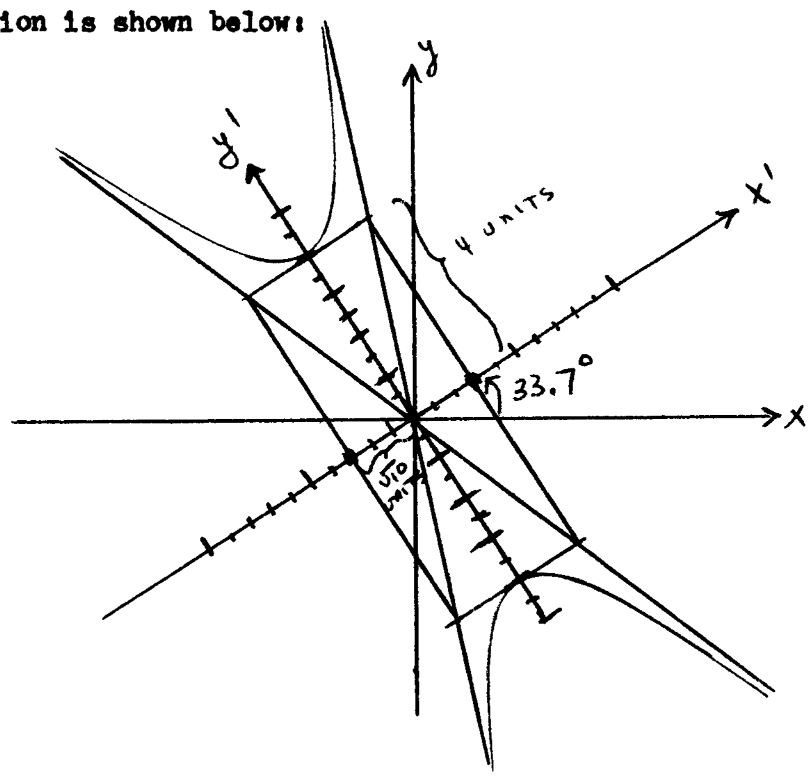
Substituting these equations into the original equation and dropping primes for convenience, we have,

$\dfrac{4(3x - 2y)^2}{13} + \dfrac{12(3x - 2y)(2x + 3y)}{13} - \dfrac{(2x + 3y)^2}{13} + 80 = 0$. This gives,

$4(9x^2 - 12xy + 4y^2) + 12(6x^2 + 5xy - 6y^2) - (4x^2 + 12xy + 9y^2) + 13 \cdot 80 = 0$, which gives,

$104x^2 - 65y^2 + 13 \cdot 80 = 0$, which reduces to $8x^2 - 5y^2 + 80 = 0$, and we therefore have,

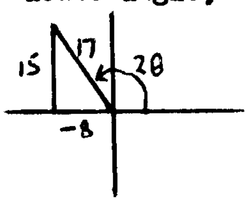
$\dfrac{x'^2}{10} - \dfrac{y'^2}{16} = -1$ (replacing primes and putting in standard form). The graph of this equation is shown below:



EXAMPLE 3: Simplify $25x^2 - 30xy + 9y^2 = 85$.

This curve is a parabola since $B^2 - AC = 0$.

Now $\tan 2\theta = -15/8$ and choosing 2θ to be in the 2nd quadrant to assure that θ is an acute angle, we have,



$\cos 2\theta = -8/17$ $\quad \cos\theta = \sqrt{\dfrac{1 - 8/17}{2}} = \dfrac{3}{\sqrt{34}}$ $\quad \sin\theta = \sqrt{\dfrac{1-(-8/17)}{2}} = 5/\sqrt{34}$

Hence, the equations of rotation are, $x = \dfrac{3x' - 5y'}{\sqrt{34}}$; $\quad y = \dfrac{5x' + 3y'}{\sqrt{34}}$.

Substituting into the original equation and dropping primes for convenience, we have:

$25(9x^2 - 30xy + 25y^2) - 30(15x^2 - 16xy - 15y^2) + 9(25x^2 + 30xy + 9y^2) = 85 \cdot 34$.

Simplifying, we obtain a degenerate parabola $y'^2 = \pm 5/2$(two straight lines).

EXERCISES(3): Reduce the following conics by means of a suitable rotation.

1. $6x^2 + 24xy - y^2 - 30 = 0$.
2. $9x^2 + 24xy + 16y^2 - 25 = 0$.
3. $2x^2 - 7xy + 2y^2 + 33 = 0$.
4. $32x^2 - 52\,xy - 7y^2 + 180 = 0$.
5. $11x^2 + 6xy + 3y^2 - 20 = 0$.

9. REDUCTION OF CONICS WITH LINEAR TERMS TO FUNDAMENTAL FORM

Sometimes we wish to translate as well as rotate so as to reduce the equation of a conic to its simplest form. This is accomplished quite readily by using the equations of translation and then using the methods shown in 8, above. Later it will be seen that using vector and matrix methods will greatly simplify the reduction involving rotation in 2-space. Also, these concepts and methods will be extended to 3-space so as to get rid of cross product and linear terms to ease the identification and plotting of curves.

Essentially there are two methods for translation. The first method involves partial differentiation and is probably the easier method. It is based on the geometrical properties of the partial derivative. Recall that the partial derivative with respect to x is obtained by treating y as a constant and differentiating the function f(x,y) with respect to x and the partial derivative with respect to y is obtained by treating x as a constant and differentiating with respect to y. Also recall that we write a function in two variables x, y as f(x,y) = 0. Thus if $f(x,y) = x^2 - 2xy - y^2 + x - y + 3$,

$$\frac{\partial F}{\partial x} = 2x - 2y + 1 \qquad \frac{\partial F}{\partial y} = -2x - 2y - 1$$

Now since the partials represent slopes of tangent lines to the given curve subject to the restrictions cited above, it is immediately seen that if we set the partials

$\frac{\partial F}{\partial x} = 0$ + $\frac{\partial F}{\partial y} = 0$, the slopes must be parallel to the respective axes when the function is implicit. Hence all that has to be done is to solve the respective equations simultaneously and the point of translation(i.e. where the tangent lines meet) is immediately obtained.

EXAMPLE 1: Determine the point of translation of the curve $9x^2 + 16y^2 - 108x + 128y + 256 = 0$ so as to obviate the x and y terms.

Here, we let $f(x,y) = 9x^2 + 16y^2 - 108x + 128y + 256 = 0$.

Hence, $\frac{\partial F}{\partial x} = 18x - 108$ $\quad \frac{\partial F}{\partial y} = 32y + 128$.

Letting the partials $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$ we obtain,

$18x - 108 = 0$ and $32y + 128 = 0$. Thus, $x = 6$ and $y = -4$.

Therefore the point of translation is $(6,-4)$. To verify, we recall that the equations of translation are $x = x' + h$, $y = y' + k$. Substituting in the original expression, we have, $9(x' + 6)^2 + 16(y' - 4)^2 - 108(x' + 6) + 128(y' - 4) + 256 = 0$. This reduces to: $9x'^2 + 16y'^2 = 324$ or if we like we may immediately render it to the form which facilitates plotting - namely, $\frac{(x - 6)^2}{36} + \frac{(y + 4)^2}{81/4} = 1$.

EXAMPLE 2: Simplify $3x^2 - 3xy + 4y^2 + 6x - 3y - 36 = 0$ by a suitable translation.

Here, we have $\frac{\partial F}{\partial x} = 6x - 3y + 6$; $\quad \frac{\partial F}{\partial y} = -3x + 8y - 3$.

Setting the partials equal to zero and solving simultaneously, we have,

$$\begin{array}{r} 6x - 3y + 6 = 0 \\ \underline{-6x + 16y - 6 = 0} \\ y \quad = 0, \end{array}$$

$x = -1$. Substituting $x = x' - 1$ and $y = y' - 0$ in to the original equation, we obtain, $3(x' - 1)^2 - 3(x' - 1)y + 4(y')^2 + 6(x' - 1) - 3(y') - 36 = 0$. This reduces to $3x'^2 - 3x'y' + 4y'^2 = 39$.

A second method utilizes the equations of translation and is slightly more tedious. Recalling that $x = x' + h$ and $y = y' + k$, we may substitute and let the x and y terms equal zero, solving for h and k.

EXAMPLE 3: Do example 1 by the second method.

Using the equation of example 1, above, we would have,

$9(x' + h)^2 + 16(y' + k)^2 - 108(x' + h) + 128(y' + k) + 256 = 0$. Rearranging terms, we get, $9x'^2 + 16y'^2 + (18h - 108)x' + (32k + 128)y' + 9h^2 + 16k^2 - 108h + 128k + 256 = 0$.

Letting $18h - 108 = 0$ and $32k + 128 = 0$, we obtain $h = 6$, $k = -4$ as before.

In 3-space this method does not lend itself to easy calculation and we resort to partial differentiation as the preferred method.

EXERCISES (4): Simplify the following by a suitable translation.

1. $4x^2 - y^2 + 24x + 8y + 16 = 0$.
2. $x^2 - 14x - 8y - 71 = 0$.
3. $x^2 - 3xy + 16x - 6y + 31 = 0$.
4. $4x^2 - 3xy + y^2 - 25x + 12y + 34 = 0$.
5. $9x^2 - 24xy + 16y^2 - 160x - 120y = 0$.

10. SIMPLIFICATION OF CONICS IN 2-SPACE WITH BOTH CROSS-PRODUCT & LINEAR TERMS

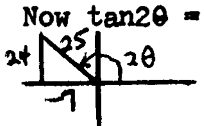
To simplify conics by both rotation and translation we just combine the above methods. In the case of the parabola(with an xy term) we should rotate first since solving the linear equations by the method of partial differentiation will give us two equations in two unknowns which will be inconsistent. In the case of the other conics, translation first is preferred, then rotation.

EXAMPLE 1: Simplify $16x^2 - 24xy + 9y^2 + 85x + 30y + 175 = 0$.

$$\frac{\partial F}{\partial x} = 32x - 24y + 85 = 0; \qquad \frac{\partial F}{\partial y} = -24x + 18y + 30 = 0.$$

These equations are inconsistent and we must rotate first(this being a parabola since $B^2 - AC = 0$).

Now $\tan 2\theta = -24/7$ and we place 2θ in the 2nd quadrant to assure an acute angle.



$\therefore \cos 2\theta = \frac{-7}{25}$. Hence $\cos\theta = \sqrt{\frac{1 - 7/25}{2}} = 3/5$; $\sin\theta = \sqrt{\frac{1 + 7/25}{2}} = 4/5$

Hence, the equations of rotation are $x = 3x' - 4y'/5$; $y = 4x' + 3y'/5$. Dropping primes for convenience and substituting, we have,

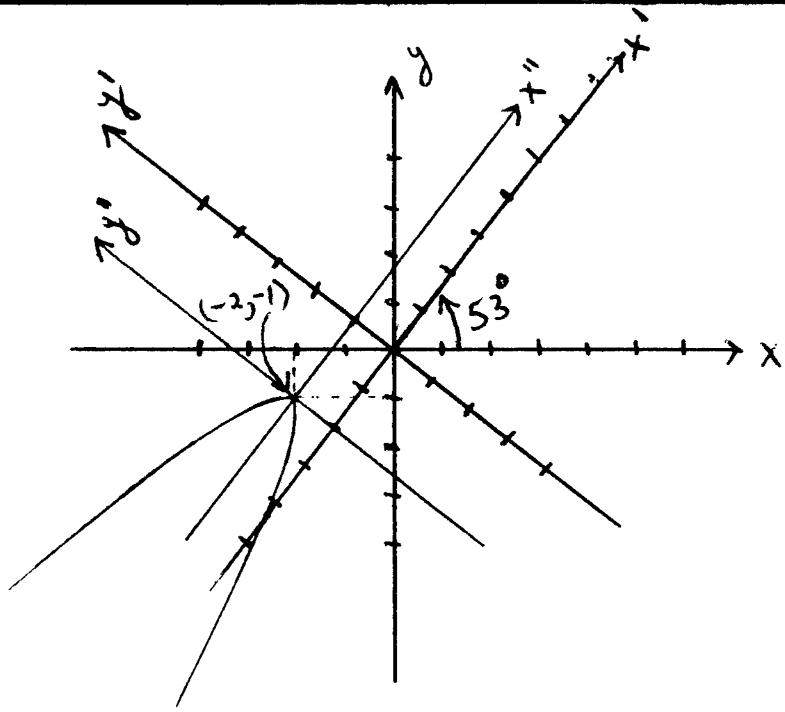
$$\frac{16(3x - 4y)^2}{25} - \frac{24(3x - 4y)(4x + 3y)}{25} + \frac{9(4x + 3y)^2}{25} + \frac{85(3x - 4y)}{5} + \frac{30(4x + 3y)}{5} + 175 = 0.$$

Hence, $16(9x^2 - 24xy + 16y^2) - 24(12x^2 - 7xy - 12y^2) + 9(16x^2 + 24xy + 9y^2)$

$+ 425(3x - 4y) + 150(4x + 3y) + 4375 = 0$. This reduces to,

$625y^2 + 1875x - 1250y + 4375 = y'^2 + 3x' - 2y' + 7 = 0$(replacing primes & reducing).

Now we apply partials to obtain: $\frac{\partial F}{\partial y'} = 10y' - 10 = 0$ which gives $y' = 1$ and substituting into $y'^2 + 3x' - 2y' + 7 = 0$, we obtain the value $x' = -2$. Substituting back into the primed equation(or we may complete the square obtaining the form, $(y' - 1)^2 = -3(x' + 2)$), we obtain the final form $(y'')^2 + 3x'' = 0$. The new origin is at $(-2,1)$ with respect to the x',y' axes and the angle of rotation is $\sin^{-1} 4/5 = 53^\circ$ approximately. The curve is a parabola and a brief sketch looks like this:



Since we rotated first, the vertex (-2,1) is with respect to the x',y' axes. If we wish to find the coordinates with respect to the x,y axes, we use,

$x = 3x' - 4y'/5 = 3(-2) - 4(1)/5 = -2$

$y = 4x' + 3y'/5 = 4(-2) + 3(1)/5 = -1$

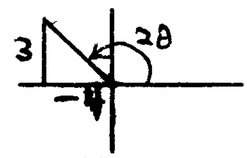
EXAMPLE 2: Simplify $5x^2 - 3xy + y^2 + 65x - 25y + 203 = 0$ and sketch briefly.

Since this represents an ellipse, we translate first. Therefore we have,

$$\frac{\partial F}{\partial x} = 10x - 3y + 65 = 0; \qquad \frac{\partial F}{\partial y} = -3x + 2y - 25 = 0.$$

Solving, we obtain, $x = -5$ and $y = 5$. Substituting $x' = x - 5$, $y' = y + 5$ and dropping primes, we have $5(x - 5)^2 - 3(x - 5)(y + 5) + (y + 5)^2 + 65(x - 5) - 25(y + 5) + 203 = 0$.

This reduces to $5x'^2 - 3x'y' + y'^2 - 22 = 0$. Next we find $\tan 2\theta = -3/4$



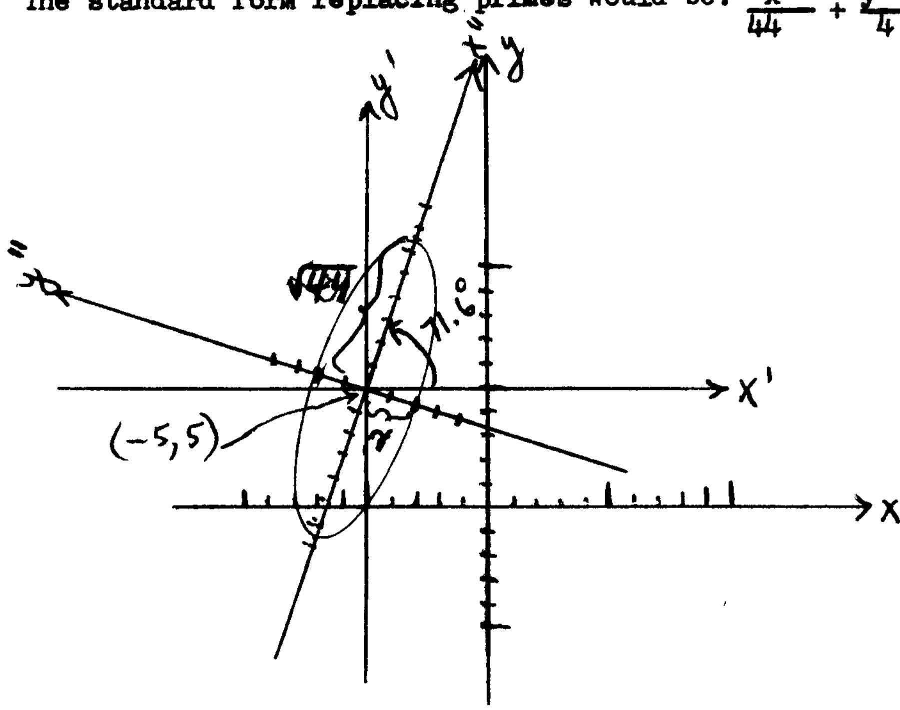
$$\cos\theta = \sqrt{\frac{1 - 4/5}{2}} = \frac{1}{\sqrt{10}}; \quad \sin\theta = \sqrt{\frac{1 + 4/5}{2}} = \frac{3}{\sqrt{10}}$$

Therefore the equations of rotation are $x' = x'' - 3y''/\sqrt{10}$; $y' = 3x'' + y''/\sqrt{10}$.

Substituting and dropping all primes for convenience we have,

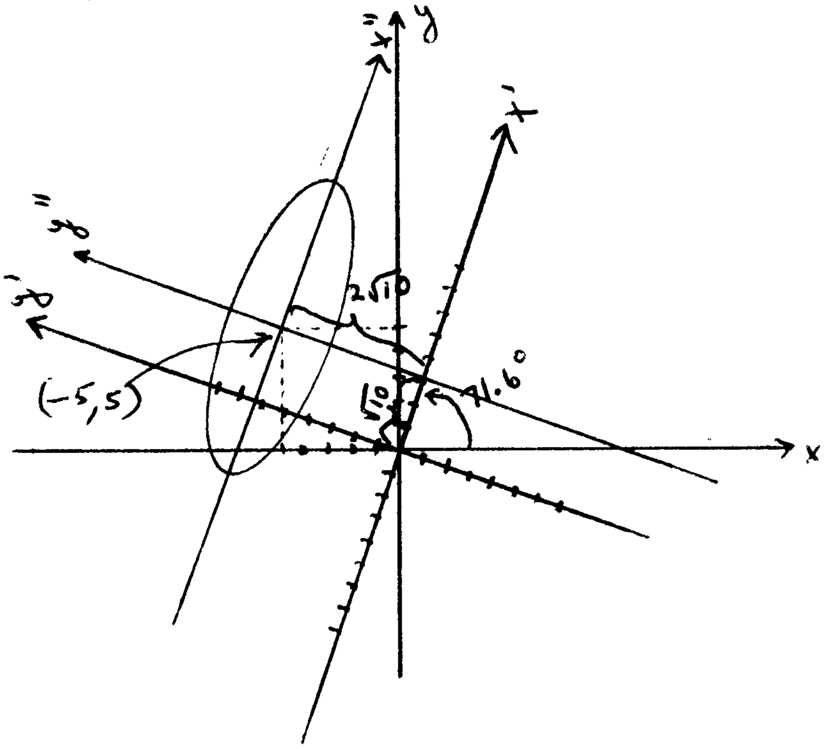
$5(x-3y)^2/10 - 3(x-3y)(3x+y)/10 + (3x+y)^2/10 - 22 = 5x^2 + 55y^2 - 220 = x^2 + 11y^2 - 44 = 0.$

The standard form replacing primes would be: $\frac{x''^2}{44} + \frac{y''^2}{4} = 1$. Briefly sketching, we have,



Note that the ellipse's major and minor axes are $\sqrt{44}$ and 2 with respect to the x'',y'' system and the center is at (-5,5) with respect to the x,y system(since translation was done first). The coordinates with respect to the x',y' system may be found from the relations $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$. When $x = -5$

and $y = 5$, we may solve for x' and y' to obtain $x' = \sqrt{10}$, $y' = 2\sqrt{10}$. This would be the answer we would have obtained had we rotated first and then translated i.e. the final diagram would have looked this way: Note the names of the axes!



EXERCISES (5): Simplify the following equations to the standard form & sketch briefly.

1. $xy = 4$.
2. $13x^2 + 12xy - 3y^2 + 40x + 30y + 10 = 0$.
3. $9x^2 + 4xy + 6y^2 + 12x + 36y + 44 = 0$.
4. $62x^2 + 168xy + 13y^2 + 380x - 90y + 575 = 0$.
5. $x^2 + 2xy + y^2 + 12\sqrt{2}x - 6 = 0$.
6. $6x^2 + 24xy - y^2 - 48x + 54y - 99 = 0$.
7. $9x^2 + 24xy + 16y^2 - 68x + 76y + 120 = 0$.
8. $-x^2 + 4xy - y^2 - 4\sqrt{2}x + 2\sqrt{2}y - 11 = 0$.
9. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0$.
10. $3x^2 - 8xy - 3y^2 + x + 17y - 10 = 0$.

CHAPTER 2 - REVIEW OF SOLID ANALYTIC GEOMETRY

11. THE GENERAL EQUATION OF THE SECOND DEGREE IN 3-SPACE

The general equation of the 2nd degree in 3 variables(in 3-space) is: $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + Gx + Hy + Iz + K = 0$. This equation represents solids known as conicoids which are analgous to the conic sections discussed in the chapter on plane analytic geometry. For example the ellipsoid(which is shaped somewhat like an egg) is analgous to the ellipse, the paraboloid will look somewhat like a parabola and so on. Again, the cross-product terms xy, yz, xz indicate that rotation is involved but in the case of the conicoids, the identification and reduction of the various conicoids is not a simple process. Translation lends itself readily to the methods of the previous chapter and for the present we shall restrict ourselves to translation by means analgous to those in the previous chapter postponing methods for getting rid of the cross-product terms to a later chapter when our knowledge of

vector and matrix methods will greatly simplify the usual tedious methods. Note that in the above equation that when $z = 0$, the equation reduces to the general equation of the second degree(i.e. a conic in 2-space) as expected.

12. STANDARD FORMS OF THE EQUATION OF A QUADRIC OR CONICOID

Since there are a number of possiblities for combinations of x,y and z in the general equation of the 2nd degree in 3-space, we must first investigate, catagorize and learn to recognize the basic conicoids in their standard form. Concomitantly, we shall also see how to sketch and recognize said forms by a simple process of allowing one variable to be zero(when we do this, we form what is called a trace of the curve) and then allowing that variable that we made zero to take on different constant values.

It must also be agreed that we shall always deal with a right-handed system so that our axes will look like Figure 15 below. The arrows represent the positive sense and we understand that we will always proceed from x to y to z in a counter-clockwise sense; all ordered triples of course, follow this convention so that the points shown in figure 16 should be self-explanatory.

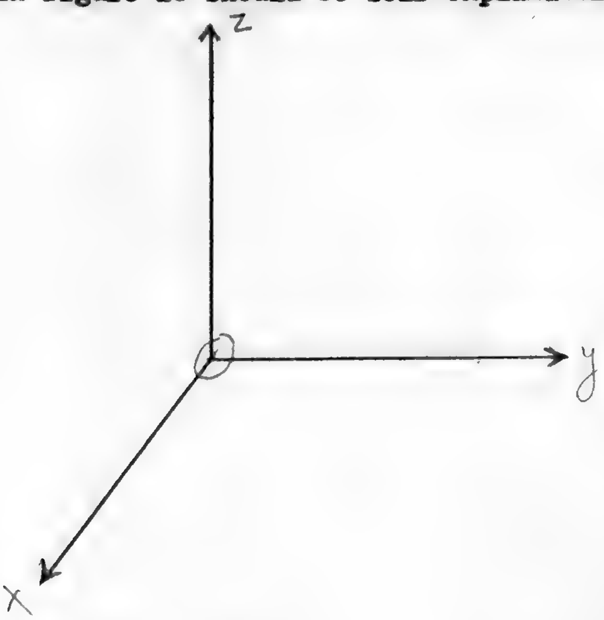


FIGURE 15

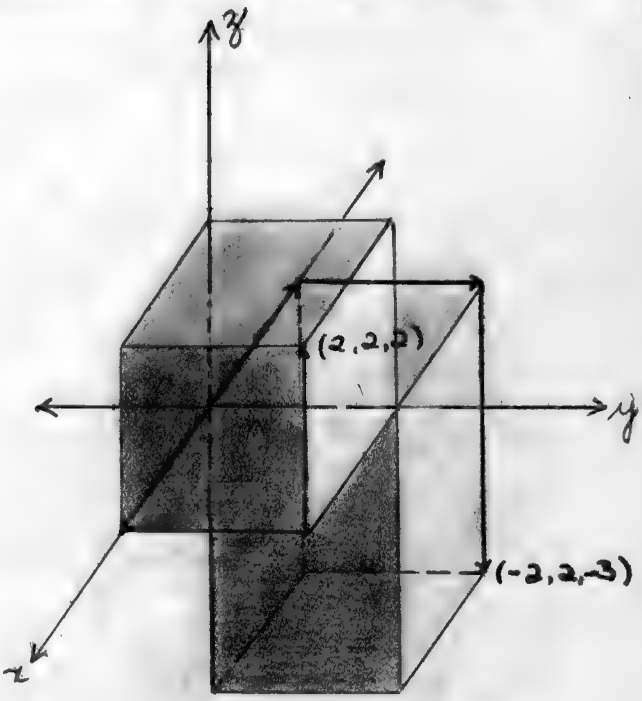


FIGURE 16

13. THE SPHERE $X^2 + Y^2 + Z^2 = R^2$

The simplest conicoid is the sphere and its equation is analgous to the 2-space equation of the circle in that the equation above represents a sphere whose center is at the origin and whose radius is R. The conicoid is shown with axes in figure 17.

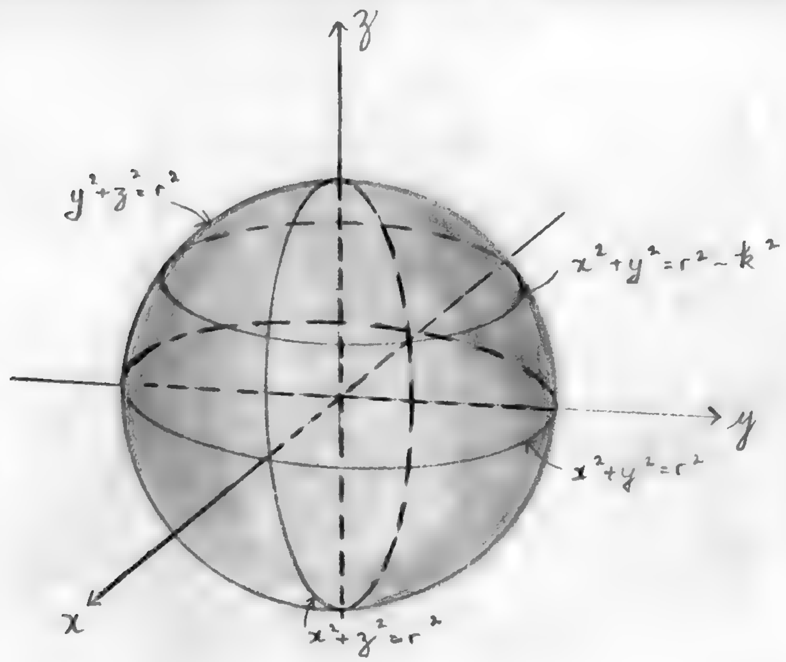


FIGURE 17

Letting x,y,z = 0 in turn, we see the 3 traces i.e. 3 circles. One will be in the yz plane when x = 0, one in the xz plane when y = 0 and one in the xy plane when z = 0. Also, if we allow, say, z to take on a constant value k, we see that as k approaches r, the radii of the circles are becoming smaller and smaller in each case. Hence the curve above becomes elementary to plot by considering each trace.

14. THE ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

The ellipsoid is analgous to the ellipse in 2-space and the 3 traces are seen to be ellipses. Also allowing each variable to assume constant values quickly verifies the graph shown.

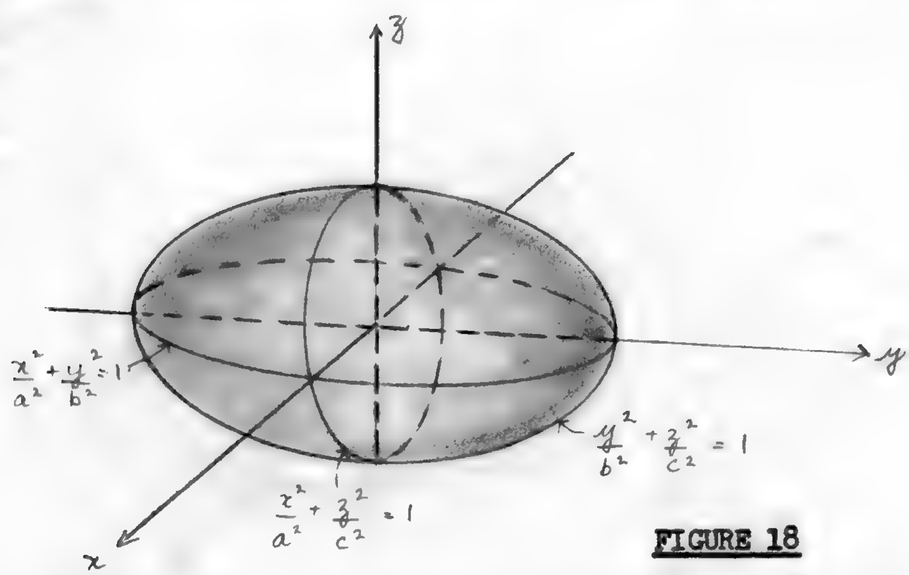
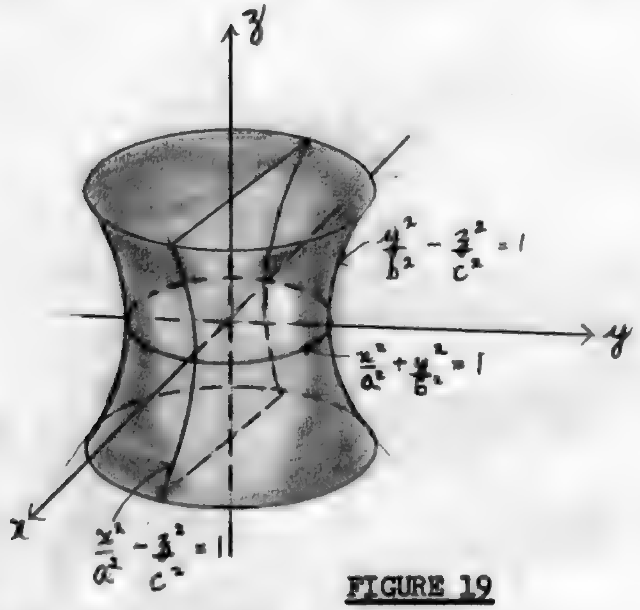


FIGURE 18

Note that the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$. represents the null set since there are no real values of x,y,z which satisfy the equation.

15. <u>THE HYPERBOLOID OF 1 SHEET</u> $\frac{X^2}{a^2} + \frac{Y^2}{b^2} - \frac{Z^2}{c^2} = 1.$

Here it is seen that we have two hyperbolas and 1 ellipse as traces. The two hyperbolas are in the xz and yz planes and the ellipse is in the xy plane. Of course, allowing the 3 varaibles to take on constant values will verify the curve shown in figure 19. Allowing the right hand side of the equation above to become negative instead of positive changes the conicoid into 16.



<u>FIGURE 19</u>

16. <u>THE HYPERBOLOID OF 2 SHEETS</u> $\frac{X^2}{a^2} - \frac{Y^2}{b^2} - \frac{Z^2}{c^2} = 1.$

The diagram is again self-explanatory. Note the 3 traces, the imaginary ellipse for x = 0 and for values of $\frac{X^2}{a^2} < 1$, when $\frac{X^2}{a^2} = 1$, then we have $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$, i.e. the point($\pm$a,0,0) satisfying the equation and so on. Note that changing the positive 1 to negative 1 above will change the curve into the form given in 15. above. Note that two of the traces are hyperbolas and the third trace is an ellipse. Note also that the curve cannot possibly pass through the origin. All of these characteristics, of course, can be gleaned from figure 20.

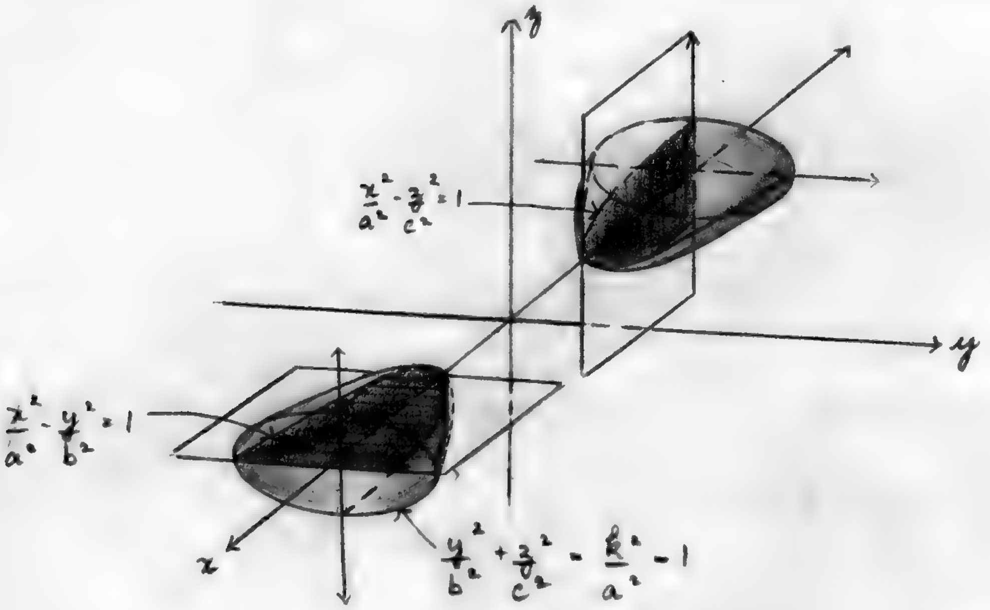


FIGURE 20

17. THE ELLIPTIC CONE $\frac{X^2}{a^2}+\frac{Y^2}{b^2}-\frac{Z^2}{c^2}=0.$

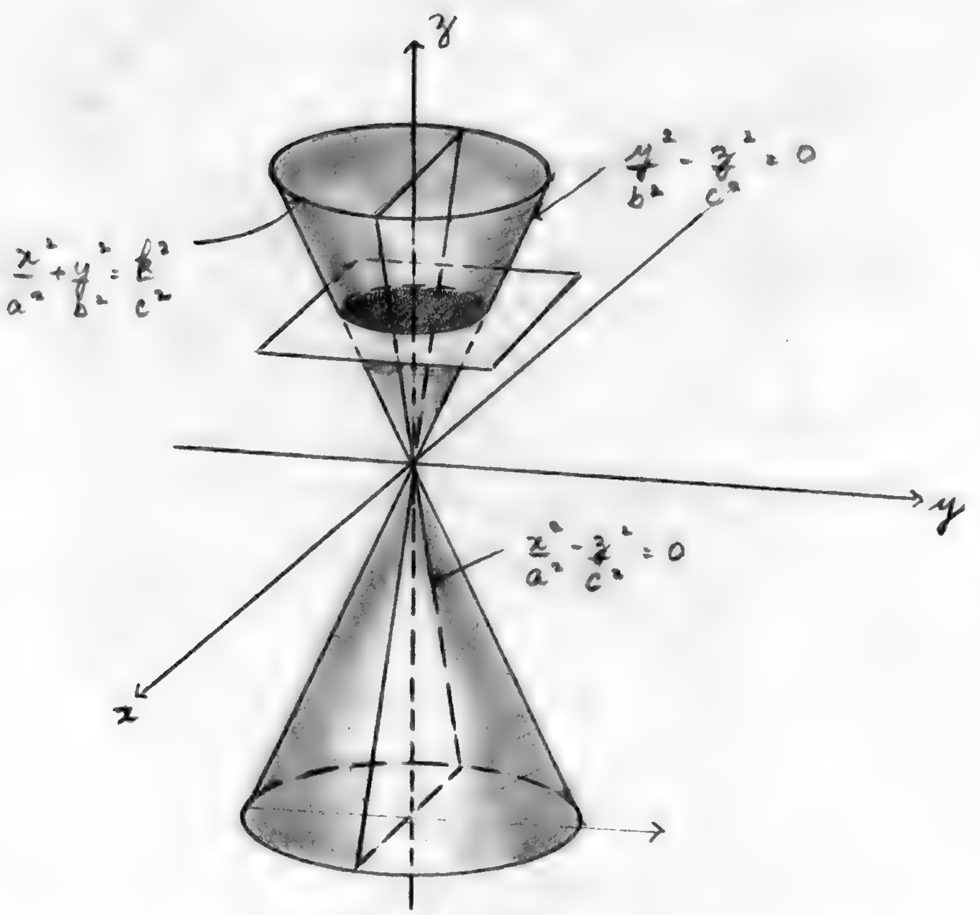
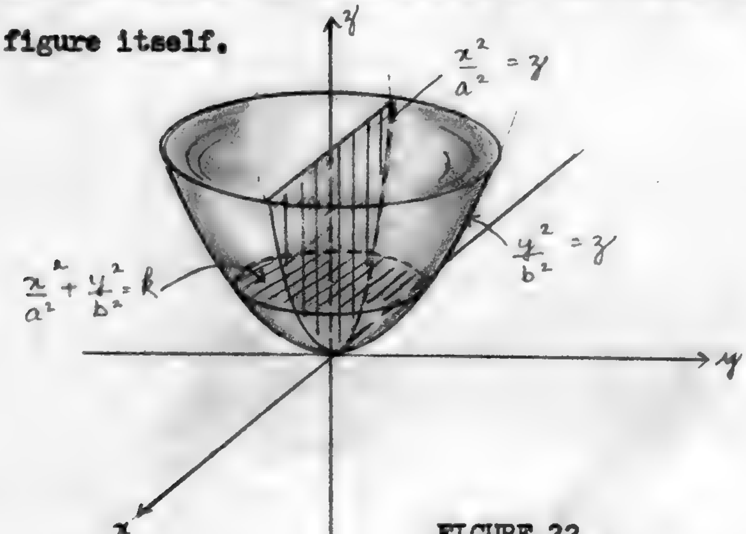


FIGURE 21

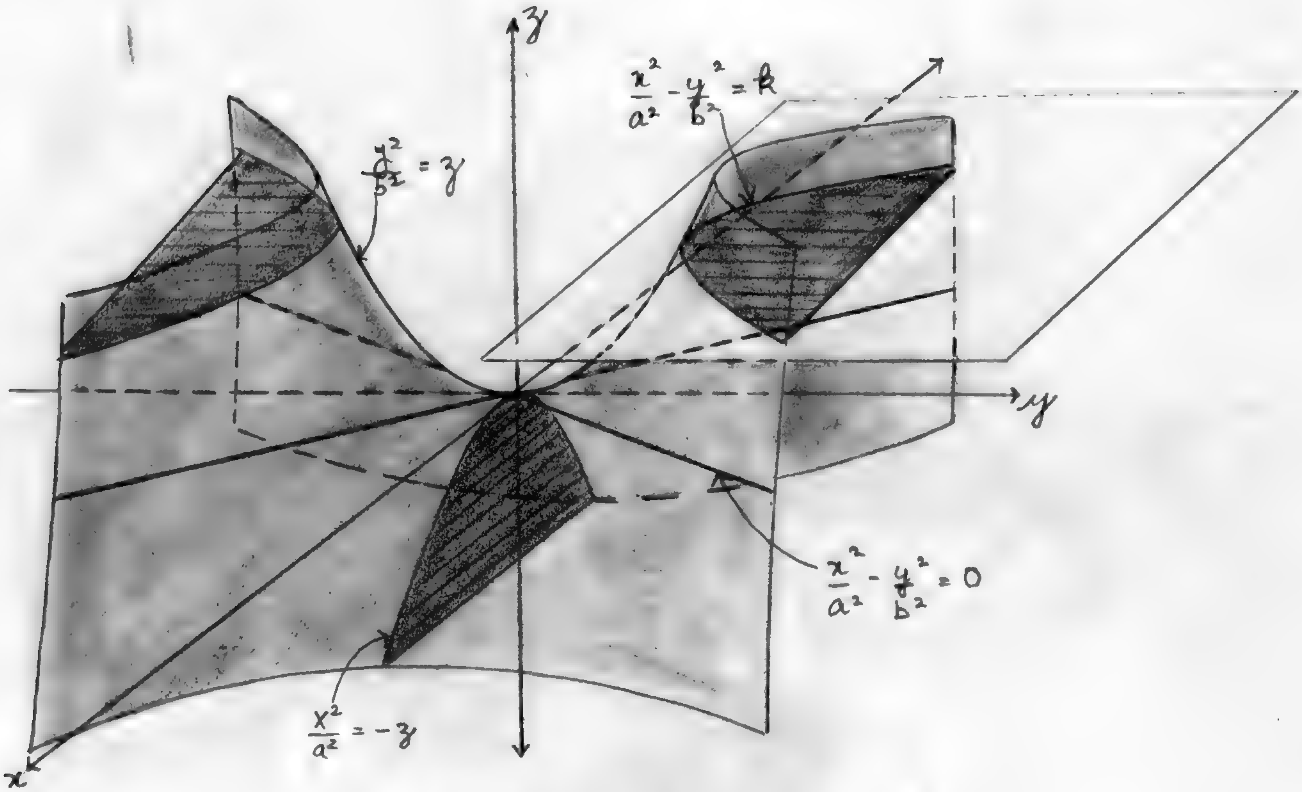
18. <u>ELLIPTIC PARABOLOID</u> $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = \pm Z.$

The figure is again self-explanatory and the negative sign on z indicates that the conicoid would open downward; other variations as in the above cases by permuting variables would only change the orientation of the figure but not alter the basic shape of the figure itself.



<u>FIGURE 22</u>

19. <u>HYPERBOLIC PARABOLOID</u> $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = \pm Z$



<u>FIGURE 23</u> $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = -z\right)$

The equations above succinctly explain the traces but it might also be noted besides the pair of intersecting lines shown in the above drawing, a surprising property of this particular surface is that for each and every point on the surface, one can always find two distinct straight lines which lie entirely on the surface.

20. THE ELLIPTIC CYLINDER $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1.$

Perhaps the easiest conicoids to draw are the cylinders. It is quickly seen from the drawing and the equation that z can take on any values. Other permutations of x,y and z, of course, do not alter the essential shape of any cylinder or of any conicoid for that matter.

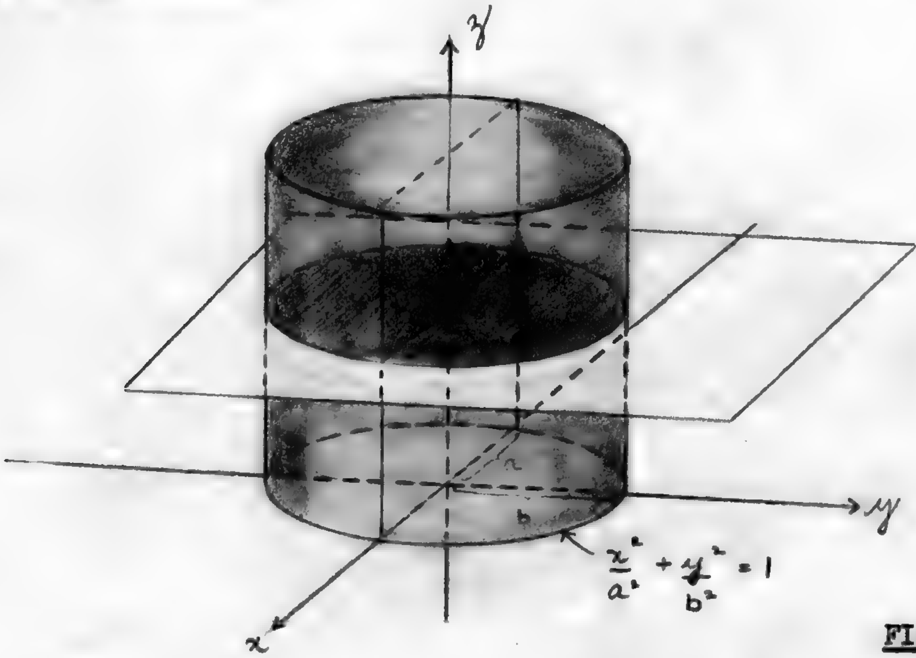


FIGURE 24

21. THE HYPERBOLIC CYLINDER $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = \pm 1.$

As in 2-space, +1 indicates that the hyperbola opens up about the x-axis and -1 indicates that the hyperbola opens up about the y-axis. These two cases are illustrated by figures 25(a) and 25(b) below:

22. THE PARABOLIC CYLINDER $Y^2 = 4pX$ or $X^2 + aY + bZ = 0.$

The two types of parabolic cylinders are shown in figures 26(a) and 26(b) below. In the case of the cylinder shown in figure 26(b), it is seen that if z is held constant, the parabola $x^2 = \pm ay$ will "move" along the y-axis according to the value of bk. Similarly, if y is held constant $x^2 = -bz$ will "move" along the z-axis according to the value of ak. When x is constant, we obtain a straight line $ay + bz + k^2 = 0$ and the values of a,b,k will determine the slope and orientation of a system of lines as shown.

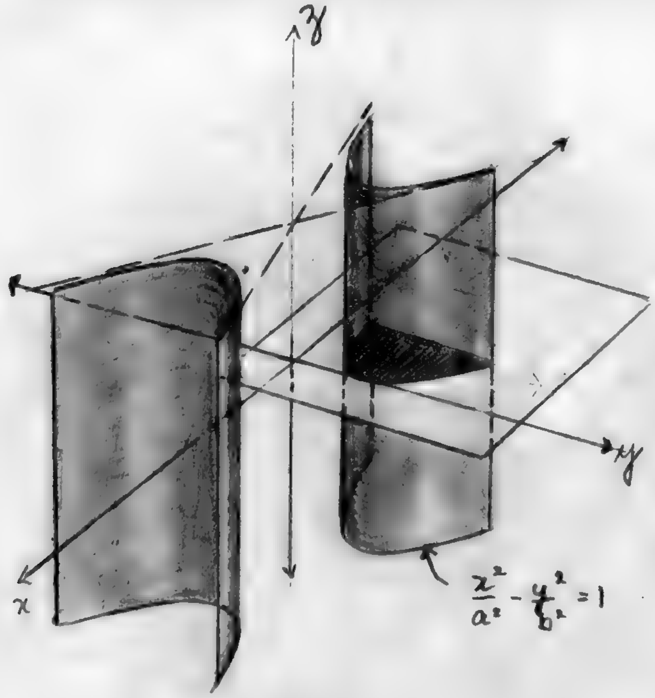


FIGURE 25(a)

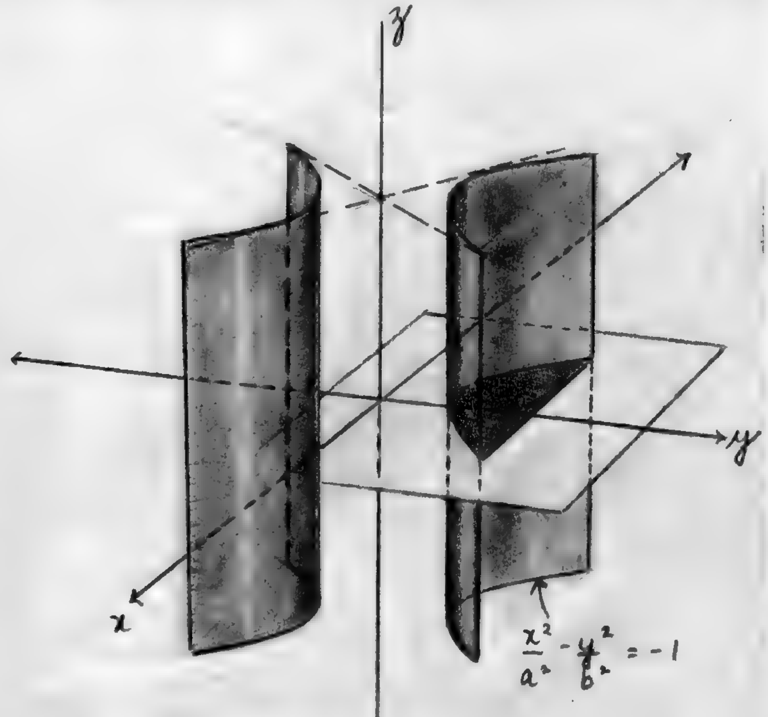


FIGURE 25(b)

FIGURE 26(a)

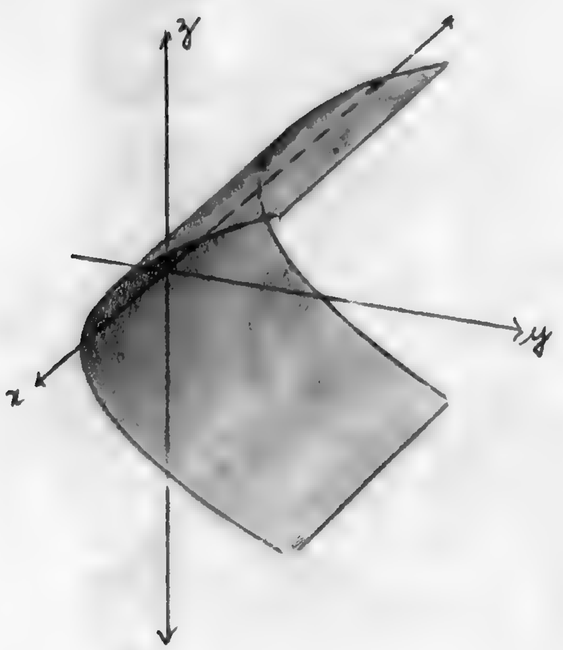
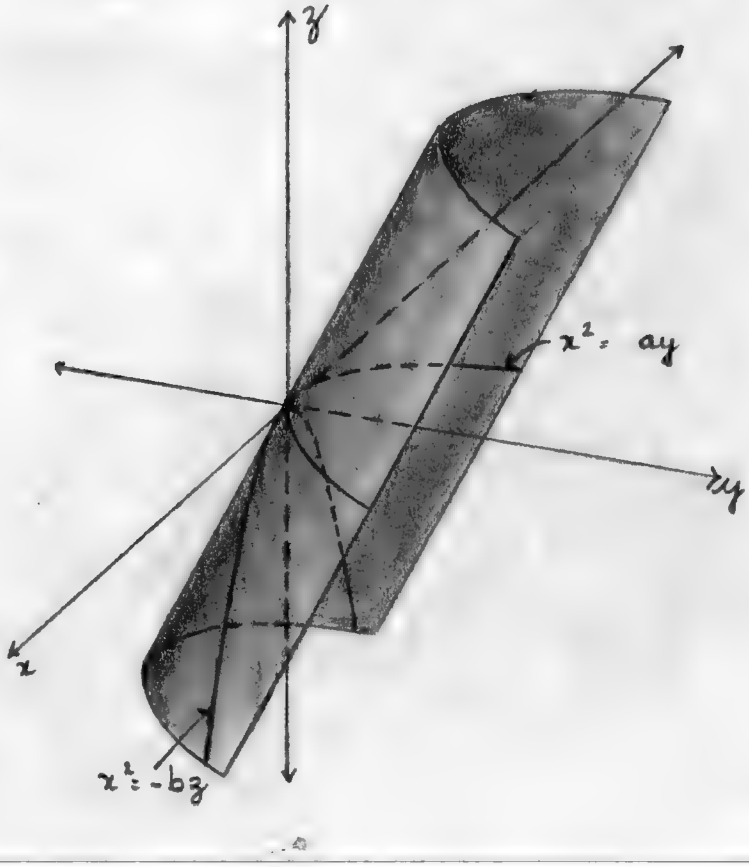
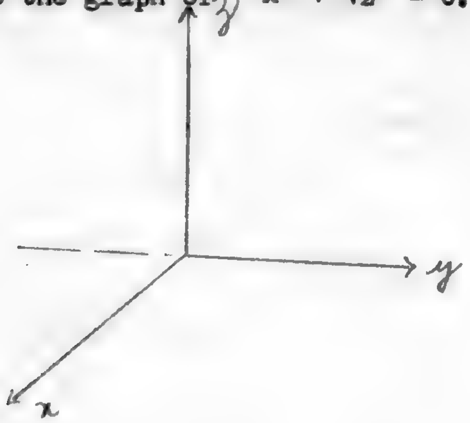


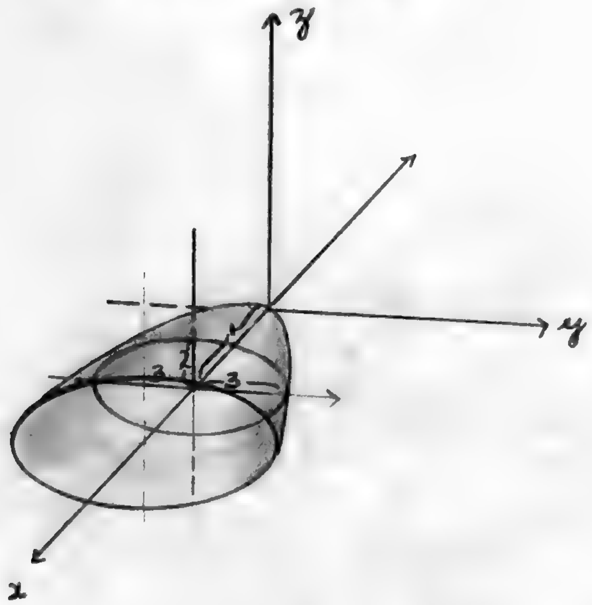
FIGURE 26(b)



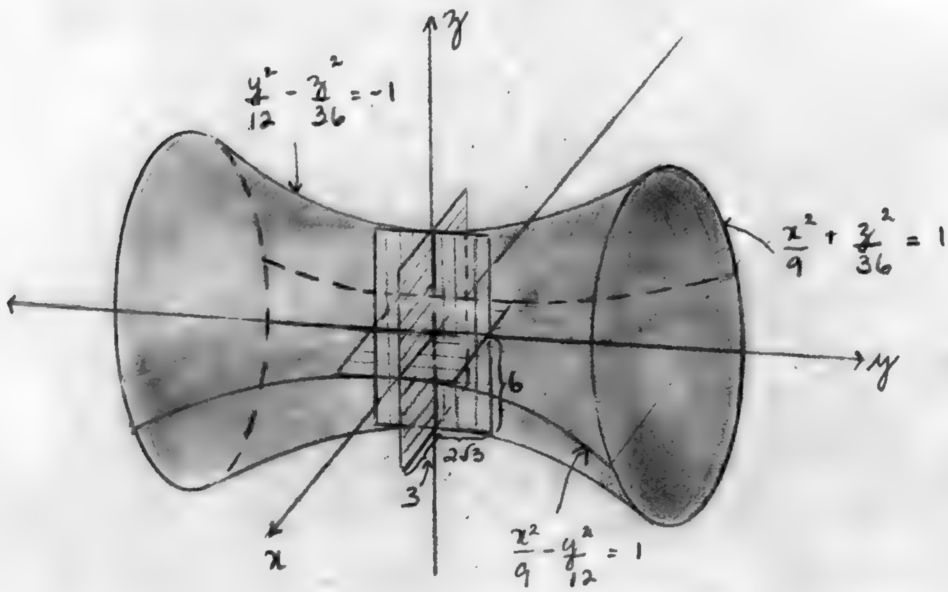
EXAMPLE 1: Plot the graph of $x^2 + 4z^2 = 0$.



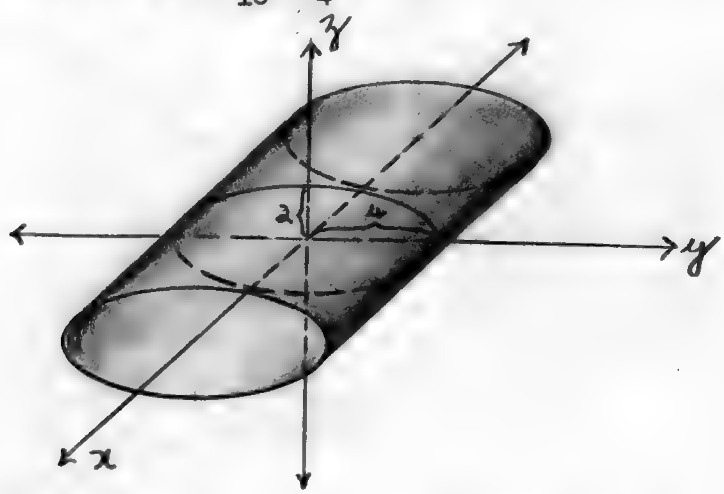
EXAMPLE 2: Plot the graph of $\frac{y^2}{9} + \frac{z^2}{4} = x$.



EXAMPLE 3: Plot the graph of $\frac{x^2}{9} - \frac{y^2}{12} + \frac{z^2}{36} = 1$.



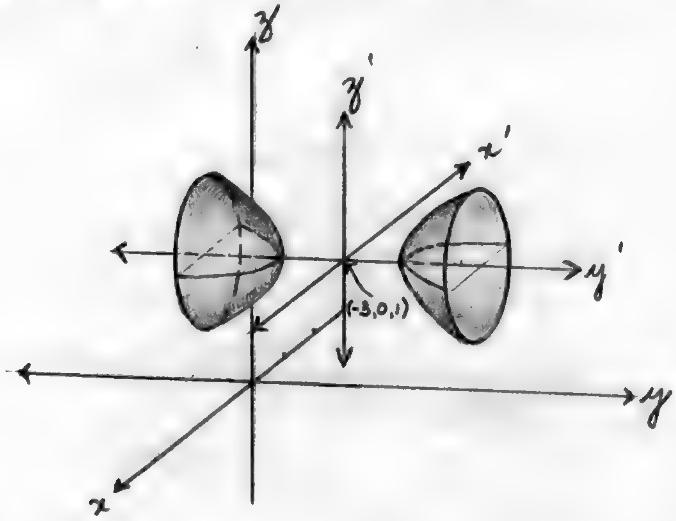
EXAMPLE 4: Plot the graph of $\frac{y^2}{16} + \frac{z^2}{4} = 1$.



EXAMPLE 5: Plot the graph of $x^2 - y^2 + 4z^2 + 6x - 8z + 14 = 0$.

We immediately complete the square to obtain: $(x + 3)^2 - y^2 + 4(z - 1)^2 = -1$. This yields:

$\frac{-(x + 3)^2}{1} + \frac{y^2}{1} - \frac{(z - 1)^2}{1/4} = 1$. This is an hyperboloid of two sheets about the y-axis which has to be translated to the new origin (-3,0,1). This is illustrated below:



EXERCISES(6): Identify the following conicoids and sketch briefly.

1. $x^2 + 9y^2 = 0$.
2. $9x^2 + 36y^2 + 4z^2 - 36 = 0$.
3. $x^2 - 9y^2 + 9z = 0$.
4. $x^2 + 4y^2 - 16z = 0$.
5. $16x^2 - 4y^2 + 4z^2 + 16 = 0$.
6. $4x^2 + 9y^2 - z^2 = 0$.
7. $4x^2 + 9y^2 - 36z^2 + 36 = 0$.
8. $x^2 - y^2 = 0$.
9. $4x^2 + 9y^2 = 36$
10. $6y^2 - 4x + z = 0$

23. REDUCTION OF CONICOIDS WITH LINEAR TERMS TO FUNDAMENTAL FORM

The equations of translation in 3-space are like those in 2-space and are easily derivable. They are: $x = x' + h;\quad y = y' + k;\quad z = z' + l.$

$$x' = x - h;\quad y' = y - k;\quad z' = z - l.$$

To simplify a quadric surface by translation, the same methods are employed that were used in Chapter 1. We may complete the square, find partial derivatives or use the equations of translation to obtain values of (h,k,l) the center of the particular conicoid in question. Again, in certain cases like those of the paraboloids we will find that sometimes the equations are inconsistent and a unique result is not always determinable. This does not make that much of a difference in 3-space as will be verified later on when reductions of conicoids is accomplished by use of vector and matrix methods.

EXAMPLE 1: Simplify the following conicoid by a suitable translation:

$x^2 + y^2 + z^2 + 2x - 6y + 22z + 115 = 0.$

Method 1: First, we see that this must represent a sphere. Second, we complete the square.

$x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 + 22z + 121 = -115 + 1 + 9 + 121 = 16.$

Hence we have, $(x + 1)^2 + (y - 3)^2 + (z + 11)^2 = 4^2$. Thus the center of the sphere is at $(-1,3-11)$ and the radius is 4.

Method 2: $\dfrac{\partial F}{\partial x} = 2x + 2;\qquad \dfrac{\partial F}{\partial y} = 2y - 6;\qquad \dfrac{\partial F}{\partial z} = 2z + 22$

Setting the partials equal to zero, we have $x = -1$, $y = 3$, $z = -11$ and the result follows.

Method 3: Let $x = x' + h;\ y = y' + k;\ z = z' + l$ and substitute into the original expression; we have,

$(x' + h)^2 + (y' + k)^2 + (z' + l)^2 + 2(x' + h) - 6(y' + k) + 22(z' + l) + 115 = 0.$

Equating the coefficients of x,y,z equal to zero yields the required result.

EXAMPLE 2: Find a suitable translation for $6x^2 + y^2 - z^2 + 12x - 6y + 4z + 5 = 0.$

Completing the square, we have, $6(x^2 + 2x + 1) + (y^2 - 6y + 9) - (z^2 - 4z + 4) = 6.$

Accordingly, we get, $\dfrac{(x + 1)^2}{1} + \dfrac{(y - 3)^2}{(\sqrt{6})^2} - \dfrac{(z - 2)^2}{(\sqrt{6})^2} = 1.$ This is an hyperboloid of one sheet with center at $(-1,3,2)$ and having $a = 1$, $b = \sqrt{6}$, $c = \sqrt{6}$.

EXAMPLE 3: Simplify $5x^2 + 7y^2 + 6z^2 - 4yz - 4xz - 6x - 10y - 4z + 7 = 0$ by a suitable translation.

Here we have cross-product terms so that we may not use the method of completing the

square. But we can either use partial derivatives or substitution by equations of translation. Using the former, we obtain,

$$\frac{\partial F}{\partial x} = 10x - 4z - 6 = 0; \quad \frac{\partial F}{\partial y} = 4y - 4z - 10 = 0; \quad \frac{\partial F}{\partial z} = 12z - 4y - 4x - 4 = 0.$$

Solving simultaneously, we obtain, $x = 1$, $y = 1$, $z = 1$. Substituting these values into the original expression, we get,

$$5(x + 1)^2 + 7(y + 1)^2 + 6(z + 1)^2 - 4(y + 1)(z + 1) - 4(z + 1)(x + 1) - 6(x + 1) - 10(y + 1) - 4(z + 1) + 7 = 0.$$ This reduces to the expression,

$$5x^2 + 7y^2 + 6z^2 - 4yz - 4xz - 3 = 0.$$

EXAMPLE 4: Simplify $4x^2 + 3y^2 + 2z^2 + 4yz - 4xy - 10x - 14y - 28z + 26 = 0$ by a suitable translation.

$$\frac{\partial F}{\partial x} = 8x - 4y - 10 = 0; \quad \frac{\partial F}{\partial y} = 6y - 4x + 4z - 14 = 0; \quad \frac{\partial F}{\partial z} = 4z + 4y - 28 = 0.$$

Solving these equations, we obtain no solution. Hence we deduce that this is a cylinder or paraboloid and we can proceed no further at this time, i.e. we must rotate first!

EXAMPLE 5: Simplify $4x^2 + 3y^2 + 2z^2 + 4yz - 4xy - 4x - 6y - 8z + 6 = 0$ by a suitable translation.

$$\frac{\partial F}{\partial x} = 8x - 4y - 4 = 0; \quad \frac{\partial F}{\partial y} = 6y + 4z - 4x - 6 = 0; \quad \frac{\partial F}{\partial z} = 4z + 4y - 8 = 0.$$

Solving, we obtain, $\begin{matrix} 2x + z = 3 \\ y + z = 2 \end{matrix}$ Let $z = 1$, then $x = 1$ and $y = 1$ and we therefore have by back substitution,

$$4(x + 1)^2 + 3(y + 1)^2 + 2(z + 1)^2 + 4(y + 1)(z + 1) - 4(x + 1)(y + 1) - 4(x + 1) - 6(y + 1) - 8(z + 1) + 6 = 0.$$ This reduces ultimately to,

$4x^2 + 3y^2 + 2z^2 + 4yz - 4xy - 3 = 0$. Note that any values of h, k, l obtained by partial differentiation will yield the same result!

EXERCISES(7): Simplify the following conicoids by a suitable translation and identify whenever possible.

1. $x^2 - 2y^2 + 3z^2 + 4x - 6z + 18 = 0.$
2. $6x^2 + 2y^2 + z^2 - 24x + 8y - 4z = 0.$
3. $x^2 - 4z^2 + 6x = 0.$
4. $x^2 + 4y^2 - 3z^2 - 2x - 12z - 11 = 0.$
5. $2x^2 + 3y^2 - 8x + 12y + 3z + 23 = 0.$
6. $2x^2 - 3y^2 - 8x + 12y + 3z + 23 = 0.$
7. $y^2 - 4x^2 + 2z - 6y - 12x + 6 = 0.$
8. $4x^2 - 4xy + 6xz - 6x - 3y^2 - 5yz + 5y + 2z^2 - 4z + 2 = 0.$
9. $7x^2 + y^2 + z^2 + 16yz + 8xz - 8xy + 2x + 4y - 40z - 14 = 0.$
10. $2x^2 + 3y^2 - 10z^2 + 20yz - 8xz - 28xy + 16x + 26y + 16z - 34 = 0.$

WRITE THE EQUATIONS FOR:

11. The sphere whose center is at (1,-2,-3) and has radius 5.
12. The hyperboloid of 1 sheet whose center is at (-1,3,0) and having its axis parallel to the y-axis where a = 1, b = 1, c = 2.
13. The hyperboloid of 2 sheets whose center is at (-1,3,0) and having its central axis parallel to the x-axis where a = 1, b = 1, c = 2.

14. The ellipsoid whose center is at $(2,-1,0)$ and semi-axes are 5, 5/4 and 5 respectively.

COMPLETE SQUARES AND LOCATE VERTICES OR CENTERS AND TELL WHAT AXIS IS THE AXIS OF SYMMETRY OR PARALLEL TO THE AXIS OF SYMMETRY.

15. $2x^2 + 3y^2 - 8x + 12y + 3z + 23 = 0.$
16. $4x^2 + 3z^2 - 4y + 12z + 12 = 0.$
17. $x^2 + 2y^2 - 3z^2 + 4x - 4y - 6z - 9 = 0.$
18. $4y^2 - 3x^2 - 6z^2 - 16y - 6x + 36z - 17 = 0.$
19. $2x^2 + 3y^2 - 6z^2 + 48z - 96 = 0.$
20. $4x^2 - 2y^2 - z^2 + 24x + 4y + 34 = 0.$

CHAPTER 3 - THE CONCEPT OF A VECTOR

24. THE DEFINITION OF A VECTOR

A vector is defined as an entity that must have both magnitude and direction. If either of these qualities is lacking then we do not have a vector. The symbol for a vector is an arrow, viz. initial point $\longrightarrow$ terminal point, the head of the arrow indicating the direction and the length of the arrow indicating the magnitude or size. Note the positions of the initial point and the terminal point. If the quality of direction is lacking in our entity whatever it might be, we refer to the entity as a scalar. It is important to distinguish between these two concepts at the outset since confusion will arise if these concepts are not firmly grasped. Also, two vectors are said to be equal if and only if their magnitudes and directions are the same!

Physically, one can think of many entities which are either vectors or scalars. For example, a common misconception in Physics is that speed is a vector. This is not the case as a little reflection will show since there is no direction implied by the term 'speed'. Thus, if one says that one is going 50mph, one is talking about a scalar quantity whereas if one says one is going 50 mph east then he is talking about a vector. This vector has a particular name - velocity.

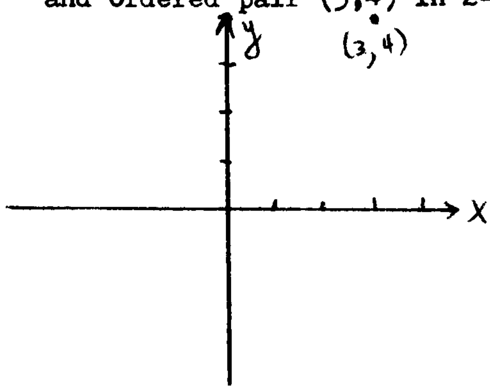
EXERCISES(8): Determine whether the following entities are scalars or vectors.

1. mass 2. kinetic energy 3. length 4. time 5. electric field intensity 6. work 7. centrifugal force 8. temperature 9. real number 10. charge 11. frequency 12. centripetal force 13. torque 14. acceleration 15. weight 16. density 17. volume 18. momentum 19. magnetic field intensity 20. calorie.

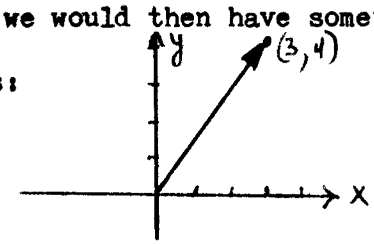
25. ALGEBRAIC AND GEOMETRIC REPRESENTATIONS OF VECTORS

The simple geometric representation of any vector as seen above is denoted by an arrow $\longrightarrow$. For convenience we may let the initial point be A and the terminal point be B. Thus the vector $A \longrightarrow B$ is denoted by $\vec{AB}$ or sometimes AB in bold face type. In this treatise bold-face type will not be used nor will the arrow above the letters where no confusion is likely. The vector whose initial

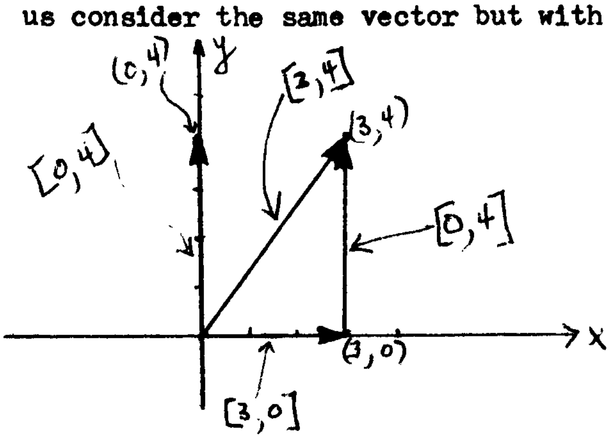
is B and whose terminal point is A will be called $\vec{BA}$ i.e. A◄——B. Because the magnitudes are the same and the directions opposite, we say therefore that $\vec{AB} = -\vec{BA}$. Sometimes a single letter will be used for representing a vector. The usual convention is to use either upper or lower case letters at the beginning of the alphabet for vectors, whilst using lower case letters such as x,y,z for scalars. We also represent the magnitude of vector $\vec{AB}$ by the symbol $|AB|$, sometimes referred to as "mag AB". Note that $|AB| = |BA|$. Let us begin by reference to a 2-space rectangular coordinate system for simplicity and extend the ideas found there to 3-space and beyond. Consider and ordered pair (3,4) in 2-space, viz.,



Now suppose that we imagine an initial point of a vector at the origin and the terminal point at (3,4); we would then have something that looked like this:



Since we are looking for a unique representation we can name the vector in terms of the coordinates in this case. We do this by use of a special notation to distinguish the vector from the ordered pair (3,4). This special notation is [3,4] where the brackets indicate that the entity is a vector - not a point! The "3" and "4" inside the brackets are called <u>components</u> of the vector. Now let us consider the same vector but with two new vectors added as shown.

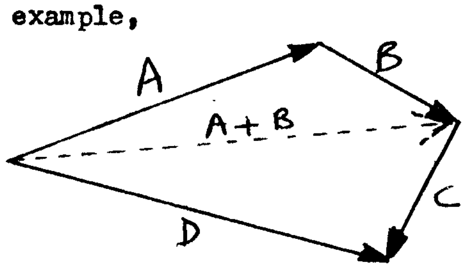


We quickly see that the name of the vector along the x-axis is [3,0] but we must reflect a little to see why the second vector is [0,4] . One must recall that the definition of a vector demanded that it have magnitude and direction only and that two vectors were equal if and only if their magnitudes and directions were both equal. Thus we may "move" the vector to the right as shown and its unique representation will not be changed one bit. It will have the same magnitude and direction. Indeed, this is the true power of vectors - by their very nature, they are <u>independent of coordinate systems and can be moved about at will as long as their magnitude and</u>

directions are preserved!

To find the magnitude of the vector $[3,4]$ above we need only to recall the pythagorean theorem which tells us that the length of $[3,4] = \left|[3,4]\right| = \left|\sqrt{3^2 + 4^2}\right| = 5$. Thus we seem to have abstracted a method for finding the components of a vector - namely, by subtracting abscissas and ordinates of two different ordered pairs. Also, the magnitude seems to be determined by taking the square root of the sum of the squares of the components of a vector. Again, referring to the above figures, one will see, by a little investigation that $[3,4] = [3,0] + [0,4]$ if we decide to define addition of two vectors by adding their respective components. This would certainly be consistent with properties of ordered pairs in geometry.

Now let us "detach" the vectors from the coordinate system and call the vector $[3,0] = OA$; $[0,4] = AB$ and $[3,4] = OB$. We can immediately see that OA + AB = OB. This suggests a possibility that any three vectors oriented in this manner might follow this rule. For example, referring to figure 26 below, we may say that the equality AB + BC = AC might be a vector law. The fact that it is indeed just that is quite easily established and another example is shown below in figure 27. This law is called the triangle law of addition. Referring back to the previous figures, we call the vector OB the <u>resultant</u> of OA and AB and we call OA and AB the <u>component vectors</u> of OB. The triangle law may be extended beyond 3 vectors by induction so that we simply break any polygon-like figure down into simpler triangle identities. For example,



Here A + B + C = D since using the law for the dotted vector yields A + B and then using the law again with C gives the required result.

In 3-space we may verify and extend the above concepts readily by reference to figure 28.

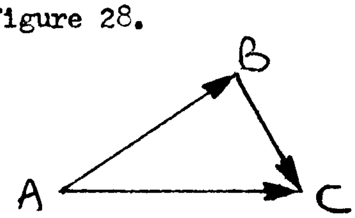


FIGURE 26

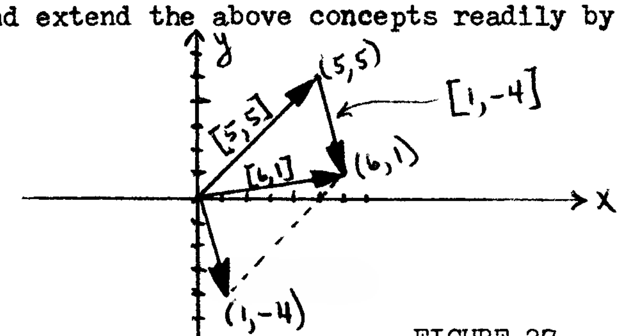


FIGURE 27

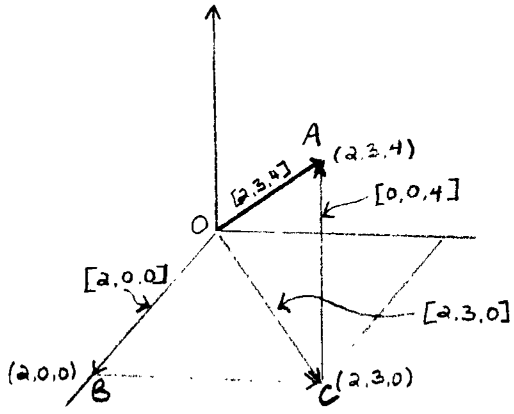
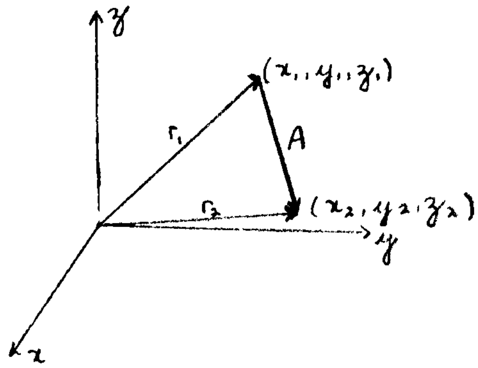


FIGURE 28

We see that OB + BC = OC, i.e. $[2,0,0] + [0,3,0] = [2,3,0]$ and that OC + CA = OA, i.e. $[2,3,0] + [0,0,4] = [2,3,4]$. Also, $|[2,3,4]| = \sqrt{4+9+16} = \sqrt{29}$ by noting that $|OB|^2 + |BC|^2 = |OC|^2$ and $|OC|^2 + |AC|^2 = |OA|^2$. Hence, $|OB|^2 + |BC|^2 + |AC|^2 = |OA|^2$.

Any vector whose initial point is at the origin in any coordinate system is called a <u>position vector</u> and obviously has the components of the particular coordinates. We can now show how to find a vector in 3-space having initial point (x_1, y_1, z_1) and terminal point (x_2, y_2, z_2). The figure below should clarify the method.



By the triangle rule $r_1 + A = r_2$. Therefore, $A = r_2 - r_1 = [x_2, y_2, z_2] - [x_1, y_1, z_1]$. This result yields the vector with components $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$. A similar proof can be constructed for 2-space.

Let us recapitulate and generalize the concepts discussed so far:

(1) $AB = -BA$ but $|AB| = |BA|$

(2) $A + B = [A_1 + B_1, A_2 + B_2, \ldots\ldots\ldots]$ (addition of components)

(3) $A = \sqrt{A_1^2 + A_2^2 + A_3^2 + \ldots\ldots\ldots}$

(4) $A - B = A + (-B) = [A_1 - B_1, A_2 - B_2, \ldots\ldots\ldots\ldots]$

Physically the addition of vectors in statics yields the so-called parallelogram law of forces and indeed historically the physical requirements engendered vector mathematics.

<u>EXERCISES(9)</u>: Find the resultant of the following vectors:

1. $[2,3] + [-1,5]$ 2. $[1,-1,3] + [5,-1,0]$ 3. $[1,-2,3] - [4,3,2] + [3,5,7]$ 4. BA + AC 5. CA + AD - BD. Find the magnitude of the vectors 6. $[-5,12]$ 7. $[-2,-1,1]$ 8. $[1,-4,-8]$ 9. $[-1,7,15] + [4,-1,7]$ 10. $[1,2,3] + [2,-5,-3] - [-1,-1,7]$. Given the points A(1,2), B(-3,4), C(-2,-3), D(1,-2), find the vectors: 11. AB 12. BA 13. BC 14. BD 15. AD - BC. Given the points A(1,2,3), B(-1,-3,-4), C(-2,3,-1), find the vectors: 16. AC 17. CA 18. BC 19. BA 20. BA - CA.

26. ALGEBRAIC PROPERTIES OF VECTORS

It can be quickly verified that vectors adhere to the usual group properties under addition - indeed they form a commutative group under addition. The four group properties for any operation "o" are:

(1) a in S and b in S implies that a o b is in S, where S is a set (closure)

(2) (a o b) o c = a o (b o c) (associativity)

(3) There exists an element e in S such that a o e = e o a = a (identity element)

(4) There exists an element a^{-1} in S such that a o a^{-1} = a^{-1} o a = e (inverse element)

If we add the property that a o b = b o a, we have a commutative group. Thus, since,

(1) A + B = B + A

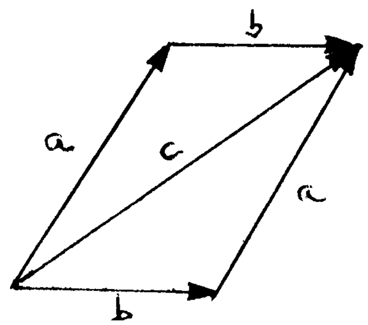
(2) A + (B + C) = (A + B) + C

(3) A + 0 = A where 0 is called the "null" vector and has components in 2-space $[0,0]$ and in 3-space $[0,0,0]$, etc.

(4) A + (-A) = 0

it is quickly seen that vectors form a commutative group under addition.

To establish that A + B = B + A for example, we can use the triangle rule in 2-space so that we have:



If we concern ourselves with the upper triangle, we see that a + b = c, whilst the bottom triangle yields b + a = c. Hence, we may conclude that a + b = b + a and commutivity is established!

Similar geometric methods may be employed to establish other additive group properties.

Since we have not defined multiplication between two vectors, we shall postpone this operation until later, but we will define properties of multiplication between scalars and vectors. Again we can verify that scalars and vectors satisfy the distributive and commutative group properties under multiplication except for the existence of an inverse element. Thus, we have:

(1) If x is in S and A is in S, then xA is in S (closure)

(2) (xy)A = x(yA) (associativity)

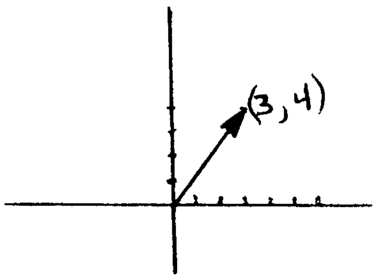
(3) There is a scalar x such that Ax = xA = A, i.e. x = 1 (identity element)

(4) $xA = Ax$ (commutivity)

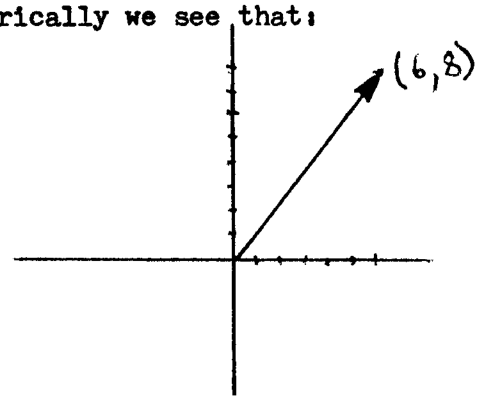
(5) $A(x + y) = Ax + Ay$ (Distributive law times/plus)

(6) $x(A + B) = xA + xB$ (Distributive law times/plus)

Thus if the above ten properties are satisfied by entities under these operations they form what is known as a linear vector space. Practically, this means that we can handle vectors almost with as much ease as elementary algebra. To see what multiplication of a vector by a scalar means we can take a very simple example. Suppose we have $A = [3,4]$, then $2A = 2[3,4] = [6,8]$. Geometrically we see that:

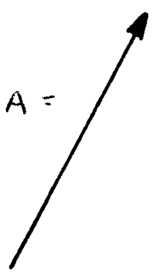


becomes



that is, the vector's magnitude has been doubled!

EXAMPLE 1: Given A = , Find 3A, -A/2

Since A = , then 3A =  & $-A/2$ = 

Note that the direction of -A/2(opposite sense) and that the vectors must also be parallel, since for any displacement one must always be able to have the vectors "overlap", so to speak. This is a fundamental property of vectors in any space, i.e. if one vector is parallel or coincident to another vector then it must be some scalar multiple of it and vice versa. To put this statement in mathematical terms, we would state that if $A \parallel B$ iff $A = tB$ where t is some scalar.

EXERCISES(10): If $A = [-2,1]$; $B = [-3,-4]$, find the resultant vectors:

1. $2A - 3B$ 2. $|A|B$ 3. $|B|A$ 4. $A/|A| + B/|B|$ 5. $2(A - 2B)$. If $A = [2,-2,1]$ $B = [4,-4,7]$, find the resultant vectors: 6. $2A - 3B$ 7. $A|B| + B|A|$ 8. $A/4 - B/2$. Find $|A/|A||$ 10. $\sqrt{|B|}$

27. UNIT VECTORS

Since we have established that if one vector is parallel or coincident to another vector then it can be expressed as a scalar multiple of that vector, it is natural to try to express any vector in terms of a vector whose magnitude is 1. Such a vector is called a unit vector

EXAMPLE 1: Express $[3,4]$ in terms of a unit vector.

First, we must unitize(or normalize as it's sometimes called) $[3,4]$ so as to make its magnitude unity. To do this, we simply divide each of its components by its magnitude. Thus we have, $u = \frac{A}{|A|} = \frac{[3,4]}{5} = [3/5, 4/5]$ and therefore $A = 5\left[\frac{3}{5}, \frac{4}{5}\right]$

Geometrically we can look upon the axes as having unit vectors lying along each axis. In 2-space, $[1,0] = i$; $[0,1] = j$. In 3-space the unit vector along the z-axis is $[0,0,1] = k$ and of course, $[1,0,0] = i$; $[0,1,0] = j$. Thus $[3,4]$ in 2-space could be expressed as $3i + 4j$ and $[1,-1,2]$ in 3-space could be expressed as $i - j + 2k$. The magnitudes of i,j and k are always 1. That is, $|i| = |j| = |k| = 1$. There is really no particular advantage in using the i,j,k system. Its historical use makes it worth mentioning and sometimes there are certain situations where precedent seemingly demands this notation although the use of the ordered pair or ordered triple notation lends itself much more readily to matrix methods which have the advantage of expediting most of the difficulties which frequently arise. The student should be able to switch from one notation to the other anyway.

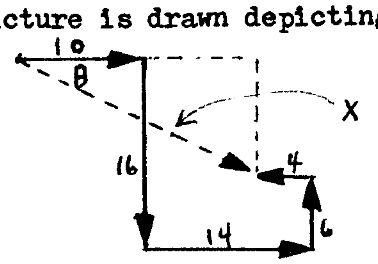
EXERCISES(11): 1. Find a unit vector parallel to $[-7,24]$ 2. Find a unit vector parallel to $[-5,-6,30]$ 3. express $2i - 3j$ in 3-space in bracket form. 4. Express $2i - 3j$ in 2-space in bracket form 5. express $[3,-5,-1]$ in terms of the unit coordinate vectors.

28. PHYSICAL PROBLEMS UTILIZING VECTORS

To illustrate some of the above concepts and to show how vectors are used in everyday physical problems, several examples are shown below. An elementary knowledge of trigonometry is assumed, of course.

EXAMPLE 1: If a man travels 10 miles east, 16 miles south, 14 miles east, 6 miles north and 4 miles west, what is the resultant displacement from the starting point?

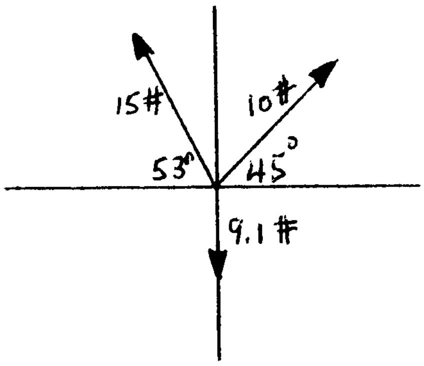
First, a picture is drawn depicting the situation as shown below:



Needless to say, if we knew the coordinates of each vector, we could then add them up to obtain the resultant. Hence, our second step is to either resolve any non-quadrantal vectors into quadrantal components(there are no non-quadrantal vectors

here) or to put all vectors into bracket form and perform the addition. Hence, we have 10 = 14 - 4 = 20 for the x-component and 16 - 6 = 10 for the y-component i.e. we obtain $[20,10]$. Now $X = |[20,10]| = 22.36$ and $\theta = \tan^{-1} y/x = 26.56^\circ$.

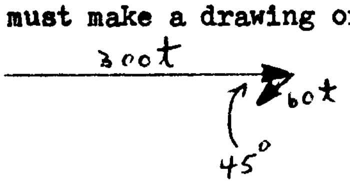
EXAMPLE 2: Find the resultant force in the figure below:



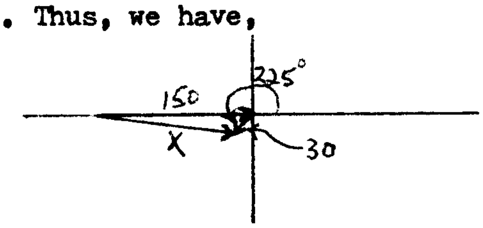
Resolving the 15lb., 10lb. and 9.1lb. vectors into their x and y components, we have, $[15\cos 127^\circ, 15\sin 127^\circ]$, $[10\cos 45^\circ, 10\sin 45^\circ]$, $[0,-9.1]$. Thus, $R = [15\cos 127^\circ, 15\sin 127^\circ] + [10\cos 45, 10\sin 45] + [0;9.1] = [9,12] + [7.1,7.1] + [0,-9.1] = [-1.9,10]$ and hence $|R| = 10.18$

EXAMPLE 3: An airplane is headed straight east at 300 miles per hour. However, the wind is simultaneously blowing the airplane southwest at 60 mph. Where is the airplane relative to its starting point after $\frac{1}{2}$ hour?

We must make a drawing of the situation at any time t. Thus, we have,



after $\frac{1}{2}$ hour we have,



Hence, we obtain $[150,0] + [30\cos 225^\circ, 30\sin 225^\circ] = X$.

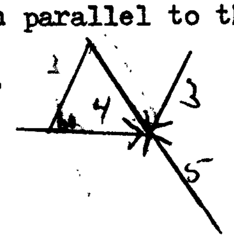
Thus, $X = [150,0] + [-21.21,-21.21] = [128.79, -21.21]$ and $|X| = 130.52$ $\theta = -9.35^\circ$

EXERCISES(12):

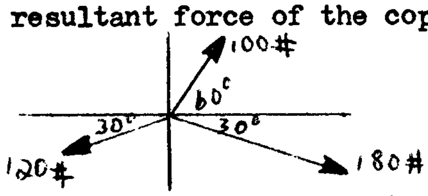
1. An airplane travels 100 miles due east and then 150 miles 60° north of west. Determine the resultant displacement.

2. The following forces act on a particle P. $F_1 = [2,5,-3]$, $F_2 = [-3,-2,0]$, $F_3 = [1,1,2]$. Find the resultant force and the magnitude of the resultant.

3. Forces of magnitudes 3,4,5 pounds act at a point in direction parallel to the side of an equilateral triangle taken in order. Find their resultant.



4. Find the resultant force of the coplanar forces shown below:



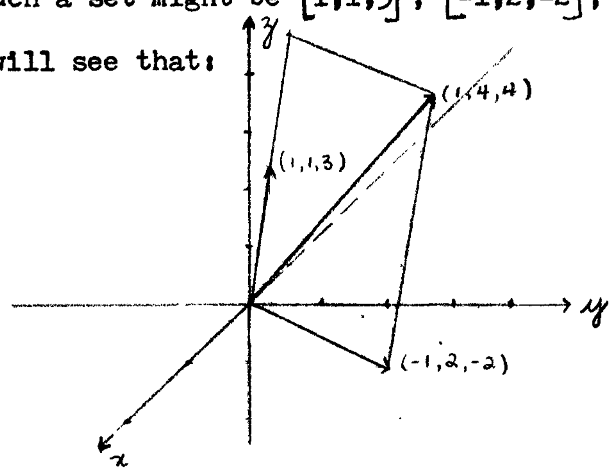
5. A boat sets out from A to go upstream across a river to C. The current of the river is 5 mph amd the speed of the boat is 10 mph. The distance from B to C is 8 miles and from Ato B is 6 miles, where B is directly across the river from A. What is the distance

which will result in the shortest time in going across upstream from A to C? HOW LONG WILL TRIP TAKE?

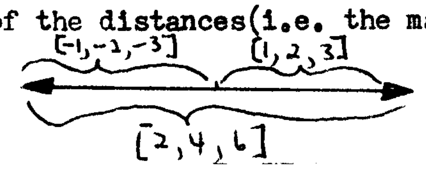
29. LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Sometimes by chance it so happens that certain vectors in 2-space can be expressed in terms of one another. For example, the vector $[6,8]$ can be expressed as $2[3,4]$. What this means geometrically is that if we were to consider the two vectors $[3,4]$ and $[6,8]$ separately, they would not indicate to us whether we had two dimensions involved since the vectors could be parallel or coincident. The term most often used for this condition is collinear i.e. the vectors lie along the same straight line even though their magnitudes differ. Also, we say that the two vectors do not span 2-space, even though they are described in two-dimensional terms.

Now let us carry this concept one step further to 3-space i.e. let us suppose that we have 3 vectors that do not span 3 space i.e. they do not occupy 3 dimensions but only 2 or 1. For convenience, let us deal with 3 vectors that are coplanar (occupying two dimensions only). Such a set might be $[1,1,3]$; $[-1,2,-2]$; $[1,4,4]$. Now if we were to plot these, we will see that:



The drawing above seems to indicate that the vectors are coplanar and indeed, $2[1,1,3] + [-1,2,-2] = [1,4,4]$ i.e. the triangle rule yields the required result. The point is that because one vector can be expressed in terms of the others we say that the vectors are linearly dependent, i.e. in 3-space they do not occupy the 3 dimensions but are either coplanar or collinear and hence one may be expressed in terms of the others. An example of 3-space collinearity would be to have the vectors $[1,2,3]$, $[2,4,6]$, $[-1,-2,-3]$. Here it is seen that $[1,2,3] = [2,4,6] + [-1,-2,-3]$ and the points are collinear since geometrically we can imagine a line in space such that the sum of two of the distances (i.e. the magnitudes of the vectors) has to be equal to the third.



Note that $[-1,-2,-3]$ is the same as $-[1,2,3]$. Transposition is possible.

Recapitulating then, we see that in 2-space vectors are linearly dependent (abbreviated L.D.) if and only if they are collinear and if we can express one vector in terms of another. In 3-space vectors are L.D. iff they are coplanar or collinear and if we can express one vector in terms of the other two. Note that by saying that we can express one vector in terms of others is the same as saying that there exist non-zero scalars such that $xA + yB + zC = 0$ in the case of 3-space, since $A = -y/xB - z/xC$ where $-y/x$ and $-z/x$ can be new scalar constants say s and t, i.e. we could just as well have $A = sB + tC$. Indeed, we can extend this concept inductively to define linear dependence in n-dimensional space. Thus, we say that a set of vectors $A_1, A_2, \ldots\ldots\ldots, A_n$ are L.D. iff there exist scalars $x_1, x_2, \ldots\ldots\ldots, x_n$ not all zero such that, $x_1A_1 + x_2A_2 + \ldots\ldots\ldots\ldots + x_nA_n = 0$.

From the examples above we can see that vectors need not be collinear or coplanar in 2 or 3-space, i.e. they may span the whole space. When this is the case vectors are not L.D., they are called <u>linearly independent</u>. Thus the vectors $[1,-1,2]$, $[1,-3,-5]$ and $[2,-1,4]$ are linearly independent(abbreviated L.I.) since not one of the vectors can be expressed in terms of the others.

There are some interesting properties regarding L.I. vectors which will be derived at this time and will prove useful in solving some practical problems. Suppose that we have two vectors $\vec{a},\vec{b}$ that are L.I., then it will be proven that if x and y are scalars and $xa + yb = 0$, it must follow that $x = y = 0$. The proof of this is quite simple and the indirect method will be used.

<u>Theorem</u>: $\vec{a},\vec{b}$, are L.I. and $x\vec{a} + y\vec{b} = 0$ implies that $x = y = 0$.

<u>Proof</u>: Suppose not, then let y, say, be $\neq 0$; then $x/y\vec{a} + \vec{b} = 0$ by hypothesis, i.e. $\vec{b} = -x/y\vec{a}$. This means that $\vec{b}$ is some scalar multiple of $\vec{a}$, but we have seen previously that it must follow that $\vec{b}$ must be coincident or parallel to $\vec{a}$ and thus $\vec{a}$ and $\vec{b}$ are L.D. - contradiction!

Now let us generalize the proof to n-space.

<u>Theorem</u>: $A_1, A_2, \ldots\ldots\ldots, A_n$ are L.I. and $x_1A_1 + x_2A_2 + \ldots\ldots\ldots + x_nA_n = 0$ imples that $x_1 = x_2 = \ldots\ldots\ldots = x_n = 0$.

<u>Proof</u>: Suppose not, then let some x, say $x_1 \neq 0$. Then $A_1 + x_2/x_1A_2 + \ldots\ldots + x_n/x_1A_n = 0$. That is, $A_1 = -x_2/x_1A_2 - x_3/x_1A_3 - \ldots\ldots\ldots\ldots - x_n/x_1A_n$.

Thus A_1 is expressed in terms of A_2, A_3, ... A_n multiplied by non-zero scalars(if the scalars are zero, then the theorem automatically follows), but if A_1 can be expressed in this fashion, the vectors must be L.D. - contradiction! Sometimes this theorem is given as the definition for L.I. vectors. To summarize again then, we say that vectors A_1, A_2, ... A_n are L.D. iff scalars x_1, x_2, ...,x_n not all zero can be found such that $x_1A_1 + x_2A_2 + \ldots\ldots + x_nA_n = 0$; vectors A_1, A_2, ..., A_n are L.I. iff $x_1A_1 + x_2A_2 + \ldots\ldots + x_nA_n = 0$ implies that $x_1 = x_2 = \ldots\ldots = x_n = 0$.

Later, we will be interested in finding L.I. vectors which are mutually perpendicular spanning 2-space and 3-space since by so doing we may place new sets of axes where we want them. Also we will develop more sophisticated techniques for determining whether vectors are L.D. or L.I. using matrices and determinants.

Right now it should be noted that any 3 vectors in 2-space must be L.D. and any 4 vectors in 3-space must also be L.D. since in either case any two of the vectors in combination with scalars must yield a third one since in each case, we have an extra vector for the dimensions involved. As soon as we can express one vector in terms of the others they are L.D.

<u>EXAMPLE 1</u>: Show that $[1,-1,2]$, $[1,-3,-5]$, $[2, -1,4]$ are L.I.

If the vectors were L.D. then there would be scalars not equal to zero, say, x,y,z such that $x[1,-1,2] + y[1,-3,-5] + z[2,-1,4] = 0$. Accordingly, we would have,

$$\left.\begin{aligned} x + y + 2z &= 0 \\ -x - 3y - z &= 0 \\ 2x - 5y + 4z &= 0 \end{aligned}\right\} \quad \text{solution is } (0,0,0), \text{ i.e. } x = 0 = y = z, \text{ Hence vectors are L.I.}$$

<u>EXAMPLE 2</u>: Show that $[1,-1,2]$, $[1,-3,-5]$, $[2,-1,4]$, $[-1,0,-1]$ are L.D.

If they are L.D., then there must exist scalars(non-zero) x,y,z,w, such that,

$$x[1,-1,2] + y[1,-3,-5] + z[2,-1,4] + w[-1,0,-1] = 0.$$

Hence, $$\left.\begin{aligned} x + y + 2z - w &= 0 \\ -x - 3y - z \quad\;\; &= 0 \\ 2x - 5y + 4z - w &= 0 \end{aligned}\right\} \quad \text{eliminating } w \text{ from (1) and (3) we have,}$$

$$\left.\begin{aligned} -x - 3y - z &= 0 \\ x - 6y + 2z &= 0 \end{aligned}\right\} \quad \text{now let } z = 9, \text{ thus,} \quad \begin{aligned} -x - 3y &= 9 \\ x - 6y &= -18 \\ -9y &= -9 \\ y &= 1 \end{aligned} \quad \text{and } \begin{aligned} x - 6 &= -18 \\ x &= -12 \end{aligned}$$

$w = x + y + 2z = -12 + 1 + 18 = 7$. Hence, we obtain,

$-12[1,-1,2] + [1,-3,-5] + 9[2,-1,4] + 7[-1,0,-1] = 0$, which is easily verified.

EXERCISES(13):

1. Show that A[2,-1], B[1,-3], C[3,2] are L.D. and find non-zero scalars x,y,z such that $xA + yB + zC \neq 0$.

2. Show that [3,4,-5], [-2,-2,3], [0,2,-1] are L.D.

3. If A = [2,-1], B = [-3,4] where A,B are L.I. and $x(A - B) + y(2A + B) + 3A = 0$, find the values of x and y.

4. If A = [-1,2,-2], B = [-3,0,-1], C = [2,2,0] are L.I. vectors and $x(A - B) + y(B - C) + z(A - C) + 3C - 3B = 0$, find the values of x,y and z.

5. Given the 3 position vectors [1,-1,2], [3,-3,3], [-3,3,0]; determine if they are L.D. or L.I.. If the former, whether they are coplanar or collinear.

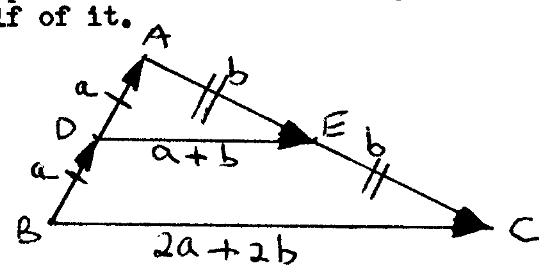
30. VECTOR ANALYSIS APPLIED TO PLANE EUCLIDEAN GEOMETRY

It would be instructive at this point to illustrate the power and usefulness of vector tools to solve problems in plane euclidean geometry.

Although there are more sophisticated techniques in some instances to solve these types of problems we will have to defer these techniques until we can build up a little more machinery via later chapters.

Essentially there are two techniques. The first utilizes the concept of L.I. and it only remains for the student to make any necessary constructions coupled with basic vector properties and a little ingenuity. The second technique requires a little knowledge of algebra and usually upon assigning vector names to certain parts of the plane figure involved, the proof practically "falls out" automatically, so to speak. The second technique will be illustrated first since it is the easier to grasp.

EXAMPLE 1: Prove that the line joining the midpoints of a triangle is parallel to the third side and equal to one-half of it.



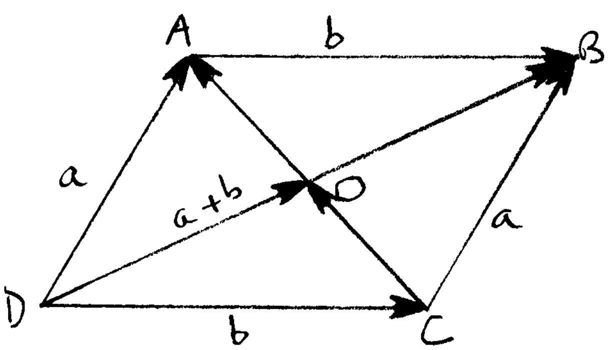
First, we draw a figure.

Second, we assign vectors as shown.

Third, we write down the vector relationships to see if any conclusions are apparent, viz.

Here it is seen that vectorially $\vec{DA} + \vec{AE} = \vec{DE}$ by the triangle rule and $\vec{BA} + \vec{AC} = \vec{BC}$. Therefore $\vec{DE} = a + b$, $BC = 2a + 2b = 2(a + b)$. Now $\vec{DE}$ is a scalar multiple of $\vec{BC}$ and hence $\vec{DE}$ is parallel to $\vec{BC}$ and since the scalar is 2 then $|DE| = \frac{1}{2}|BC|$ QED:

EXAMPLE 2: Prove that the diagonals of a parallelogram bisect each other.



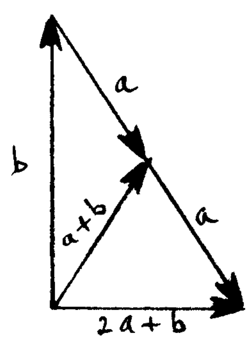
Let $\overrightarrow{DO} = x\overrightarrow{DB} = x(a + b)$

Let $\overrightarrow{CO} = y\overrightarrow{CA} = y(a - b)$

Now $b + \overrightarrow{CO} = \overrightarrow{DO}$, i.e. $b + y(a - b) = x(a + b)$.

Hence, $a(x - y) + b(x + y - 1) = 0$, but a,b are L.I. and thus, $x - y = 0$; $x + y - 1 = 0$ which gives $x = y = \frac{1}{2}$; QED!

EXAMPLE 3: Prove that the median to the hypotenuse of a right-angled triangle is $\frac{1}{2}$ of it.



Must show that $|a + b| = |a|$. Now considering a to have components $[a_1, a_2]$ and $b = [b_1, b_2]$, we have $|a + b| = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$ and $|a| = \sqrt{a_1^2 + a_2^2}$. Therefore, we must show that: $(a_1 + b_1)^2 + (a_2 + b_2)^2 = a_1^2 + a_2^2$ or the equivalent statement $b_1^2 + b_2^2 + 2a_1b_1 + 2a_2b_2 = 0$.

But, because we have a right-angled triangle, we therefore have:
$|2a + b|^2 + |b|^2 = |2a|^2$ or in terms of the respective components, we have, $(2a_1 + b_1)^2 + (2a_2 + b_2)^2 + b_1^2 + b_2^2 = 4a_1^2 + 4a_2^2$. This reduces to the following relation after a little elementary algebra: $b_1^2 + b_2^2 + 2a_1b_1 + 2a_2b_2 = 0$. QED!

EXERCISES(14):

1. Prove that the lines joining the mid-points of the sides of a quadrilateral form a parallelogram.
2. Prove that the medians of any triangle trisect each other.
3. Show that the angle bisectors of any triangle meet at a point.
4. Prove that the median from 1 vertex to the opposite side in any parallelogram trisect the diagonal that they intersect.
5. If O is any point within triangle ABC and P,Q,R are the mid-points of the sides AB, BC, CA respectively, prove that OA + OB + OC = OP + OQ + OR.
6. If one pair of opposite sides of a quadrilateral are equal, the mid-points of the other 2 sides and the mid-points of the diagonals form a rhombus.
7. If the perpendiculars from two of the vertices of a triangle on the opposite sides are equal, the triangle is isosceles.
8. If two medians of a triangle are equal, the triangle is isosceles.
9. The sum of the perpendiculars drawn from any point within an equilateral triangle to the 3 sides is equal to the perpendicular drawn from the vertex to the base. (hint: use problem 5).

10. Show that if two parallelograms have a common diagonal the other angular points are at the corner of another parallelogram.

CHAPTER 4 - MATRICES AND DETERMINANTS

Unfortunately we need some more tools before we are able to extend our knowledge of vectors. We will need to know something about matrices as mentioned in chapter 2 in order to facilitate rotation in 2 and 3-space. Also, it might have been noted in chapter 3, section 29 that a knowledge of matrices would have helped since we were really dealing with systems of simultaneous linear equations. A working knowledge of determinants and matrices is essential then if we are to develop more sophisticated vector techniques. In this effort we are not going to concern ourselves with proofs of theorems that we use since any good text on Matrices and determinants would provide these. Instead, the use of the definitions and theorems will be stressed - especially how they may be used with vector analysis.

31. DEFINITION OF DETERMINANTS

The first entity with which we deal is a square array of symbols, usually numbers, called a determinant. Rectangular arrays of symbols are also possible but these are called matrices(singular-matrix). The chief difference is that while there is a numerical or other type of value associated with a determinant, there need not be if the entity is a square matrix! Thus, the student should not confuse a square matrix with a determinant. The symbols indicating that an array of symbols is a determinant are two bars. Thus,

$$\begin{vmatrix} a & x & -1 \\ 1 & 3 & -3 \\ 2 & 4 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \quad \text{are determinants.}$$

In the first determinant above, the three symbols a,x,-1 are called a row, whilst a,1,2 are called a column. Furthermore identification is given to each element by reference to its position in terms of rows and columns. Thus x would be the element in the 1st row and 2nd column, 2 would be in the 3rd row, 1st column, etc. The first determinant is referred to as a 3 x 3 and the second as a 2 x 2. Mathematicians use double subscripts to indicate each element's position and to represent an n x n determinant, we would have,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{vmatrix}$$

Note that the 1st subscript indicates the row and the 2nd subscript indicates the column.

Another entity associated with each determinant is an entity called a <u>minor</u> or <u>cofactor</u>. These are determinants obtained by striking out the ith row and jth column of any element in the original determinant, the sign of the cofactor being determined by whether i + j is even or odd. If it is even, it's positive, if odd, then it's negative. The symbols used for cofactors are usually upper case letters associated with the elements of the determinant. Thus A_{12} would be the sub-determinant found by striking out the 1st row and 2nd column and its sign would be negative.

<u>EXAMPLE 1</u>: Find the cofactors of the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 7 \\ 0 & -2 & 3 \end{vmatrix} \quad A_{11} = \begin{vmatrix} 5 & 7 \\ -2 & 3 \end{vmatrix} \quad A_{12} = -\begin{vmatrix} -4 & 7 \\ 0 & 3 \end{vmatrix} \quad A_{13} = \begin{vmatrix} -4 & 5 \\ 0 & -2 \end{vmatrix} \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ -4 & 7 \end{vmatrix}$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \quad A_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}$$

$$A_{33} = \begin{vmatrix} 1 & 2 \\ -4 & 5 \end{vmatrix}$$

<u>EXAMPLE 2</u>: Find A_{24} of the following determinant:

$$\begin{vmatrix} 1 & 3 & 4 & 5 & 7 \\ 6 & 3 & 2 & 5 & 1 \\ -7 & -2 & -1 & 0 & -1 \\ 8 & 7 & 6 & -2 & 3 \\ 4 & -6 & 2 & -3 & 1 \end{vmatrix} \qquad A_{24} = \begin{vmatrix} 1 & 3 & 4 & 7 \\ -7 & -2 & -1 & -1 \\ 8 & 7 & 6 & 3 \\ 4 & -6 & 2 & 1 \end{vmatrix}$$

32. <u>EVALUATION OF DETERMINANTS</u>

There are a variety of techniques for evaluating a determinant but they are all predicated on the definition of the evaluation. The motivation for evaluating the determinant in this definition was, of course, profoundly influenced by geometric properties but need not concern us here. At this juncture we will define the evaluation and then develop more rapid techniques for this evaluation strictly as an intellectual exercise. The value of the determinant is usually indicated by the

symbol $\triangle$ (delta) and $\triangle = a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} - \ldots\ldots + (-1)^{n-1}a_{n1}A_{n1}$.

This can be expressed in short form by $\sum_{i=1}^{i=n}(-1)^{i-1}a_{i1}A_{i1}$. (Note here that the sign of the cofactors is automatically taken into consideration!)

Thus, if $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ Then $\triangle = a_{11}a_{22} - a_{21}a_{12}$, since $A_{11} = a_{22}$ and $A_{21} = -a_{12}$.

Again, if

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ then } \triangle = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Therefore $\triangle = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$.

<u>EXAMPLE 3</u>: Evaluate: $\begin{vmatrix} -1 & 2 & -5 \\ 3 & 6 & -2 \\ 4 & 3 & -9 \end{vmatrix}$

$$= -1\begin{vmatrix} 6 & -2 \\ 3 & -9 \end{vmatrix} - 3\begin{vmatrix} 2 & -5 \\ 3 & -9 \end{vmatrix} + 4\begin{vmatrix} 2 & -5 \\ 6 & -2 \end{vmatrix} = 48 + 9 + 104 = 161.$$

<u>EXAMPLE 4</u>: Show that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ represents a straight line in 2-space.

$$= x(y_1 - y_2) - x_1(y - y_2) + x_2(y - y_1) = x(y_1 - y_2) + y(x_2 - x_1) + x_1y_2 - x_2y_1 = 0$$

i.e. $Ax + By + C = 0$ which represents a straight line in 2-space.

One can readily see that evaluation of determinants larger than 3 x 3 can become a very tedious process. Fortunately, there are some theorems(whose proofs we will omit) which obviate much of the tedium of evaluation. These theorems are now listed and examples will be given utilizing the conclusions of these theorems.

<u>Theorem 1</u>: If the corresponding rows and columns of D are interchanged the value of D is unchanged.

<u>EXAMPLE 5</u>: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$ and $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$.

<u>Theorem 2</u>: If any two rows(or columns) of D be interchanged, $\triangle$ changes sign.

<u>EXAMPLE 6</u>: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$ <u>BUT</u> $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2$.

<u>Theorem 3</u>: If any two rows(or columns) of D are alike or in proportion, then $\triangle = 0$.

EXAMPLE 7: $\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0$ and $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 2 & -5 \end{vmatrix} = 0.$

THEOREM 4: If each row(or column of D be multiplied by k, then the value of D is $k\triangle$.

EXAMPLE 8: $\begin{vmatrix} -1 & 2 & -5 \\ 3 & 6 & -2 \\ 4 & 3 & -9 \end{vmatrix} = 161$ (from example 3) and $\begin{vmatrix} -2 & 4 & -10 \\ 3 & 6 & -2 \\ 4 & 3 & -9 \end{vmatrix} = 322$ (2 times row 1).

Theorem 5: If to each element of a row(or column) of D is added k times the corresponding element in another row(or column), $\triangle$ is unchanged.

This is an extremely useful theorem since it allows us to put zeroes in propitious places.

EXAMPLE 9: $\begin{vmatrix} -1 & 2 & -5 \\ 3 & 6 & -2 \\ 4 & 3 & -9 \end{vmatrix} = \begin{vmatrix} -1 & 2 & -5 \\ 0 & 12 & -17 \\ 4 & 3 & -9 \end{vmatrix}$ keeping row 1 fixed and multiplying it by 3 and adding it to row 2.

$= \begin{vmatrix} -1 & 2 & -5 \\ 0 & 12 & -17 \\ 0 & 11 & -29 \end{vmatrix}$ keeping row 1 fixed and multiplying it by 4 & adding it to row 3. $= -1\begin{vmatrix} 12 & -17 \\ 11 & -29 \end{vmatrix} = 161.$

Theorem 6: $\triangle = \sum_{i=1}^{n} (-1)^{i-1} a_{ni}A_{ni}$ (no sum on n) $= \sum_{i=1}^{n} (-1)^{i-1} a_{in}A_{in}$ (no sum on n). This says that we may use any row or column to evaluate D.

EXAMPLE 10: Evaluate $\begin{vmatrix} -1 & 2 & -5 \\ 3 & 6 & -2 \\ 4 & 3 & -9 \end{vmatrix}$ by using column 2.

We have, $-2\begin{vmatrix} 3 & -2 \\ 4 & -9 \end{vmatrix} + 6\begin{vmatrix} -1 & -5 \\ 4 & -9 \end{vmatrix} -3\begin{vmatrix} -1 & -5 \\ 3 & -2 \end{vmatrix} = 38 + 174 - 51 = 161.$

Theorem 7: If any row(or column) of D is zero, then $\triangle = 0$.

EXAMPLE 11: Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ -5 & 2 & 3 \end{vmatrix} = 0.$

EXERCISES(15): Find the value of the following determinants. Use theorems 1-7 whenever possible.

1. $\begin{vmatrix} 2 & -5 \\ 3 & 6 \end{vmatrix}$

2. $\begin{vmatrix} 3 & -6 & 2 \\ 1 & 0 & -5 \\ -2 & 3 & -4 \end{vmatrix}$ 3. $\begin{vmatrix} 2 & -5 & 3 \\ 5 & 6 & -2 \\ -4 & 2 & 1 \end{vmatrix}$ 4. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ 5. $\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$

6. $\begin{vmatrix} \sin\theta & \cos\theta & \sec\theta \\ \csc\theta & \tan\theta & \csc\theta \\ \cot\theta & \sin\theta & \cot\theta \end{vmatrix}$ 7. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ 8. $\begin{vmatrix} 1 & 2 & 5 & -7 \\ -3 & 2 & -3 & 4 \\ -5 & 6 & -7 & 8 \\ -2 & 1 & 0 & 6 \end{vmatrix}$ 9. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$

10. $\begin{vmatrix} -1 & 5 & 2 & 3 & -2 \\ -2 & 1 & 3 & 4 & -2 \\ 3 & -5 & 2 & 6 & -1 \\ 4 & -2 & 6 & -3 & -1 \\ -2 & 1 & 3 & -4 & 1 \end{vmatrix}$

33. DEFINITION OF A MATRIX

It was seen that a square array of symbols together with an evaluation technique yielded a determinant. A matrix is rather like an extension of this concept. It is a rectangular array(not necessarily square) of symbols(usually numbers) but it differs from the determinant in that there is no value assigned to the array as was the case when dealing with determinants. Instead the whole array is treated as an element of a set and the set elements have certain operations defined on them. Thus, they form an algebra - called matrix algebra and we will be interested in a lot of the properties of this algebra. The symbol for a matrix is a pair of brackets [ ] and as will be seen, is not entirely a coincidence with our choice of bracket use for vectors. Indeed, as will be seen later, a vector can be considered as a special type of matrix, i.e. a column matrix. The matrix is always described in terms of rows first and columns second. Thus, a 2 x 3 matrix(read "two by three") means 2 rows and 3 columns whereas a 3 x 2 matrix would be 3 rows and 2 columns. Some examples of matrices are shown below:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ $\begin{bmatrix} 4 & \sin\theta \\ -5 & 6 \\ 3 & \cos\theta \end{bmatrix}$

<u>2 x 2</u> <u>3 x 1</u> <u>2 x 3</u> <u>3 x 2</u>

Sometimes one is interested in finding non-zero determinants of matrices. This will tell one the <u>rank</u> of the matrix and the rank of the matrix will yield information about systems of equations, for example. A <u>zero matrix</u> is one that contains all zeroes and its rank is said to be zero. The rank of a matrix is formally defined as r if at least one of its r-square minors is different from 0, whilst every r + 1 square minor(if any) is zero.

<u>EXAMPLE 1</u>: Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Here $\triangle = 0$, but $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$

Thus, $r = 2$.

If the determinant of any <u>square matrix</u> = 0, it is called <u>singular</u>.

<u>EXAMPLE 2</u>: Find the rank of $\begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix}$ $\triangle = 0$ and $A_{ij} = 0$ $(i,j = 1,2,3)$, Thus $r = 1$.

34. <u>OPERATIONS WITH MATRICES</u>

Addition and subtraction are possible with matrices if and only if they are the same type - sometimes they are said to be <u>conformable</u>. Addition and subtraction are defined in a manner sim-ilar to addition and subtraction of vectors - namely, the elements or components of each are either added or subtracted to one another.

<u>EXAMPLE 1</u>: Add: $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 7 \end{bmatrix} + \begin{bmatrix} -3 & 2 & 0 \\ -3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1-3 & 2+2 & 3+0 \\ -4-3 & 5-2 & 7-1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 3 \\ -7 & 3 & 6 \end{bmatrix}$

<u>EXAMPLE 2</u>: add: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -3 \end{bmatrix}$ = NOT POSSIBLE!

<u>EXAMPLE 3</u>: add: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ -5 & 4 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 8 & 0 \\ 11 & 3 \end{bmatrix}$

Multiplication is a little more complicated and division is not defined ex-cept in terms of multiplication by an inverse element where division by the zero matrix is excluded, of course. Multiplication of two matrices is performed by us-ing the <u>row</u> of elements of the left multiplier and combining these with the <u>column</u> elements of the right multiplier. After this the results of the multiplications are added together to give the corresponding answer element. Some simple examples

should suffice to elucidate this concept.

EXAMPLE 4: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -5 & 6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1(-5) + 2(-2) & 1(6) + 2(3) \\ 3(-5) + 4(-2) & 3(6) + 4(3) \end{bmatrix} = \begin{bmatrix} -9 & 12 \\ -23 & 30 \end{bmatrix}$

EXAMPLE 5: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$

EXAMPLE 6: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -2 & 3 \\ 4 & 5 \\ -7 & -2 \end{bmatrix}$ = NOT POSSIBLE!

EXAMPLE 7: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & -5 & 3 \\ 6 & -9 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -23 & 11 \\ 30 & -51 & 25 \\ 46 & -79 & 39 \end{bmatrix}$

It will be noticed that multiplication is not possible whenever we have an $m \times n$ matrix times a $p \times q$ matrix where $n \neq p$, i.e. we must have an $m \times n$ matrix times an $n \times p$ matrix and furthermore the answer matrix must be an $m \times p$!

EXERCISES(16): If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} -3 & 4 & 5 \\ -6 & 2 & 10 \\ 9 & -1 & -15 \end{bmatrix}$

1. What is the rank of A,B,C? 2. Find A + B 3. Find BA + C 4. $|BA + C|$ 5. AB

35. TYPES OF MATRICES

There are various types of matrices with which one must become familiar for subsequent work and to facilitate manipulation. It was found that the division and multiplication of poloynomials were made extremely less cumbersome by "detaching" coefficients from the variables concerned and working with the coefficients alone. This process, it will be recalled, was termed "synthetic" division. The use of matrices is somewhat analgous to this process and again coefficients are detached and put into an array called a matrix so that they may be manipulated with much more expediency than they would otherwise. Now the types of matrices with which we will be concerned are:

(1) Transpose matrix
(2) Diagonal matrix
(3) Identity matrix
(4) Equivalent Matrix
(5) Symmetric matrix
(6) Adjoint matrix
(7) Inverse matrix

It must be understood that there are more types of matrices than these, but we really need not be concerned about them here. Also, we will again assume the general properties of these types of matrices without proof and we will utilize these theorems or properties in specific examples. Let us deal with the definitions of the above matrices in the order given and give specific examples of each.

(1) <u>TRANSPOSE OF A MATRIX</u>: A transpose of a matrix A is defined to the matrix obtained by interchanging rows and columns of A. Its symbol is A'.

<u>EXAMPLE 1</u>: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

<u>EXAMPLE 2</u>: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

There are two important theorems regarding transposes:

<u>Theorem 1</u>: The transpose of the sum of 2 matrices is the sum of the transposes, i. e. $(A + B)' = A' + B'$.

<u>EXAMPLE 3</u>: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -3 \\ 2 & -3 \\ -1 & -3 \end{bmatrix}$ $A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ $B' = \begin{bmatrix} -1 & 2 & -1 \\ -3 & -3 & -3 \end{bmatrix}$

$A + B = \begin{bmatrix} 0 & -1 \\ 5 & 1 \\ 4 & 3 \end{bmatrix}$ and $(A + B)' = \begin{bmatrix} 0 & 5 & 4 \\ -1 & 1 & 3 \end{bmatrix}$

<u>Theorem 2</u>: The transpose of a product of two matrices is the product of their transposes <u>in reverse order</u>! i.e. $(AB)' = B'A'$

<u>EXAMPLE 4</u>: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -5 & 18 \end{bmatrix}$ and $\begin{bmatrix} -2 & 6 \\ -5 & 18 \end{bmatrix}' = \begin{bmatrix} -2 & -5 \\ 6 & 18 \end{bmatrix}$

Now $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}' = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 6 & 18 \end{bmatrix}$

(2) <u>DIAGONAL MATRIX</u>: This is defined to be a square matrix with diagonal elements only where all other elements are zeroes. The symbol of a diagonal matrix is diagA. Thus,

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ are diagonal matrices.

(3) <u>IDENTITY MATRIX</u>: This matrix is defined to a a diagonal matrix whose diagonal elements are always equal to 1. Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ are identity matrices.

The motivation for calling these types of matrices identity matrices is readily seen if one recalls the third property of a multiplicative group which requires an element I in the set(in this case, the elements are themselves matrices) to exist such that A•I = A. The identity matrix I does satisfy this requirement as a simple example will show; thus,

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Obviously, diagonal and identity matrices do not exist AS non-square matrices!

(4) <u>EQUIVALENT MATRIX</u>: This is defined to that matrix B which is obtained from another matrix A by a sequence of elementary transformations. Elementary transformations are like those manipulations performed on determinants when evaluating them. They consist of interchanging rows or columns, multiplication or division of any row or column by any number whatever except zero and the multiplication of a particular row or columnn by a suitable scalar and then adding the result to another row or column.

Specifically, we are more interested in row transformations rather than column transformations since we will be dealing with systems of linear equations. Examples of equivalent matrices are shown below. Above each tilde(indicating a transformation) is written the particular technique employed.

<u>EXAMPLE 5</u>: Find an equivalent matrix to: $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix} = A.$

$$\begin{bmatrix}1&2&0&-1\\3&4&1&2\\-2&3&2&5\end{bmatrix}\overset{-3R_1+R_2}{\sim}\begin{bmatrix}1&2&0&-1\\0&-2&1&5\\-2&3&2&5\end{bmatrix}\overset{2R_1+R_3}{\sim}\begin{bmatrix}1&2&0&-1\\0&-2&1&5\\0&7&2&3\end{bmatrix}\overset{C_2\leftrightarrow C_3}{\sim}\begin{bmatrix}1&0&2&-1\\0&1&-2&5\\0&2&7&3\end{bmatrix}$$

$$\underset{2C_2+C_3}{\sim}\begin{bmatrix}1&0&2&-1\\0&1&0&5\\0&2&11&3\end{bmatrix}\sim \text{ etc.}$$

All of the above matrices are equivalent to A. If this process is continued, eventually, one ends ups with a diagonal matrix with 1's or 0's only. This is the so-called <u>normal form</u>, but need not concern us here.

<u>EXAMPLE 6</u>: Show that $\begin{bmatrix}1&1&2&1\\-1&-3&-1&0\\2&-5&4&-5\end{bmatrix}$ is equivalent to: $\begin{bmatrix}1&0&0&-6\\0&1&0&1\\0&0&1&3\end{bmatrix}$

$$\begin{bmatrix}1&1&2&1\\-1&-3&-1&0\\2&-5&4&-5\end{bmatrix}\underset{-2R_1+R_3}{\overset{R_1+R_2}{\sim}}\begin{bmatrix}1&1&2&1\\0&-2&1&1\\0&-7&0&-7\end{bmatrix}\underset{R_3\div -7}{\sim}\begin{bmatrix}1&1&2&1\\0&-2&1&1\\0&1&0&1\end{bmatrix}\underset{R_2\leftrightarrow R_3}{\sim}\begin{bmatrix}1&1&2&1\\0&1&0&1\\0&-2&1&1\end{bmatrix}$$

$$\underset{-2R_2+R_3}{\sim}\begin{bmatrix}1&1&2&1\\0&1&0&1\\0&0&1&3\end{bmatrix}\underset{-R_2+R_1}{\sim}\begin{bmatrix}1&0&2&0\\0&1&0&1\\0&0&1&3\end{bmatrix}\underset{-2R_3+R_1}{\sim}\begin{bmatrix}1&0&0&-6\\0&1&0&1\\0&0&1&3\end{bmatrix}$$

We will see subsequently that equivalent matrices provide an expedient method for solving systems of linear equations.

(5) <u>SYMMETRIC MATRIX</u>: This is a square matrix A where A = A'- that is, where the elements about the diagonal are the same. Thus,

$A = \begin{bmatrix}1&2&3\\2&-1&4\\3&4&-2\end{bmatrix}$ is symmetric since A = A'.

(6) <u>ADJOINT MATRIX</u>: This is a square matrix A where the elements are made up of the co-factors of the determinant of A such that the cofactors are written in transpose order. Thus, if $A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ then adj $A = \begin{bmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{bmatrix}$

EXAMPLE 7: Find the adjoint of $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 2\\3 & 3 & 4\end{bmatrix}$. Now $A_{11} = \begin{bmatrix}3 & 2\\3 & 4\end{bmatrix}$ $A_{12} = -\begin{bmatrix}2 & 2\\3 & 4\end{bmatrix}$ etc.

Thus, $A_{11} = 6$; $A_{12} = -2$; $A_{13} = -3$; $A_{21} = 1$; $A_{22} = -5$; $A_{23} = 3$; $A_{31} = -5$; $A_{32} = 4$; $A_{33} = -1$.
Hence, adj A = $\begin{bmatrix}6 & 1 & -5\\-2 & -5 & 4\\-3 & 3 & -1\end{bmatrix}$

(7) INVERSE OF A SQUARE MATRIX: This is a square matrix A^{-1} such that when mutiplied with A, it gives the identity matrix I. That is, $AA^{-1} = I = A^{-1}A$. Notice that this satisfies the fourth multiplicative group postulate.

EXAMPLE 8: Show that $\begin{bmatrix}1 & -3 & 2\\-3 & 3 & -1\\2 & -1 & 0\end{bmatrix}$ is the inverse of: $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{bmatrix}$. By multiplication,

$$\begin{bmatrix}1 & -3 & 2\\-3 & 3 & -1\\2 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & -3 & 2\\-3 & 3 & -1\\2 & -1 & 0\end{bmatrix}$$

Important theorems relating to inverse matrices are the following:

Theorem 3: The determinant of A^{-1} is the inverse of determinant A, i.e. $|A^{-1}| = \frac{1}{|A|}$

EXAMPLE 9: Let A = $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & -1\end{bmatrix}$ Then $A^{-1} = \begin{bmatrix}-1.7\dot{6} & .\dot{8}6 & -.0\dot{9}\\1.\dot{5}3 & -.\dot{7}3 & .1\dot{9}\\-.\dot{0}9 & .\dot{1}9 & -.0\dot{9}\end{bmatrix}$ $\frac{1}{|A|} = \frac{1}{30} = |A^{-1}|$

Theorem 4: The inverse of the transpose is the transpose of the inverse; $(A')^{-1} = (A^{-1})'$.

EXAMPLE 10: $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^{-1} = \dfrac{\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}}{2} = \begin{bmatrix}2 & -1\\-3/2 & \frac{1}{2}\end{bmatrix}$ where A = $\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$

Now $A^{-1} = \dfrac{\begin{bmatrix}4 & -3\\-2 & 1\end{bmatrix}}{2} = \begin{bmatrix}2 & -3/2\\-1 & \frac{1}{2}\end{bmatrix}$ and $(A^{-1})' = \begin{bmatrix}2 & -1\\-3/2 & \frac{1}{2}\end{bmatrix}$ which agrees with the above!

Theorem 5: The inverse of the product of two matrices A,B is the product of the inverses in REVERSE ORDER! i.e. $(AB)^{-1} = B^{-1}A^{-1}$.

Incidentally, although we do not need to find inverses of nonsquare matrices, the above theorems yield a method for finding an inverse. The required relationship is, $A^{-1} = (A'A)^{-1}A'$ since $(A'A)^{-1} = A^{-1}(A')^{-1}$, then $(A'A)^{-1}A' = A^{-1}(A')^{-1}A'$, but $(A')^{-1}A' = I$. This is predicated upon the assumption

that an inverse exists for the square matrix A'A. For example,

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 1 & 4 & 1 & 3 \\ 1 & 3 & 5 & 4 \end{bmatrix} \frac{\begin{bmatrix} 17 & -9 & -5 \\ -4 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For detailed example, refer to appendix 1.

36. SUMMARY OF ALGEBRAIC PROPERTIES OF MATRICES

(1) Addition and subtraction of matrices is not possible unless the matrices have the same number of rows and columns.

(2) Multiplication of matrices is possible only when the number of columns of the first matrix are the same as the number of rows of the second matrix.

(3) Non-singular square matrices form a commutative group under addition.

(4) " " " " " group under multiplication.

(5) The commutative law does not hold for non-singular square matrices under multiplication but if we define multiplication of a matrix A by a constant k, we mean each element is to be multiplied by this constant. Thus $2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$.

(6) Matrices form a vector space since the 10 conditions in chapter 3, section 26 are satisfied.

37. METHODS FOR FINDING THE INVERSE OF A NON-SINGULAR SQUARE MATRIX

1st Method: $A^{-1} = \dfrac{\text{adj } A}{|A|}$ (adjoint method).

Example 1: Find A^{-1} if $A = \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}$ We have, $A_{11} = -4$; $A_{12} = -5$; $A_{21} = -3$; $A_{22} = -2$. Also Det $A = -7$.

Therefore, $A^{-1} = \dfrac{\begin{bmatrix} -4 & -3 \\ -5 & -2 \end{bmatrix}}{7} = \begin{bmatrix} 4/7 & 3/7 \\ 5/7 & 2/7 \end{bmatrix}$ To check we see that,

$$\begin{bmatrix} 4/7 & 3/7 \\ 5/7 & 2/7 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 4/7 & 3/7 \\ 5/7 & 2/7 \end{bmatrix}$$

EXAMPLE 2: Find A^{-1} if $A = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ Det $A = -7$

$A_{11} = 3$; $A_{12} = 2$; $A_{13} = 2$; $A_{21} = -5$; $A_{22} = -8$; $A_{23} = -1$; $A_{31} = -1$; $A_{32} = -3$; $A_{33} = -3$.

Therefore, $A^{-1} = \dfrac{\begin{bmatrix} 3 & -5 & -1 \\ 2 & -8 & -3 \\ 2 & -1 & -3 \end{bmatrix}}{-7}$ Check: $\dfrac{\begin{bmatrix} 3 & -5 & -1 \\ 2 & -8 & -3 \\ 2 & -1 & -3 \end{bmatrix}}{-7}\begin{bmatrix} -3 & 2 & -1 \\ 0 & 1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \dfrac{\begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}}{-7} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Notice it is easier to work with matrices with integers as elements instead of fractions. Thus, it is usually a good idea to "carry along" the determinant denominator of the fraction as in example 2.

<u>2nd Method</u>: (augmented matrix method)

This method utilizes the row operations of finding an equivalent matrix to an augmented matrix formed by attaching the unit matrix to the original matrix and then changing the original matrix into a unit matrix. This method is particularly adaptable to computer techniques since the number of memories required to find the inverse is equal to the number of elements plus the number of memories required to do the operations of addition, subtraction and multiplication(usually 3).

<u>EXAMPLE 3</u>: Find A^{-1} if $A = \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}$ We form the sugmented matrix $\begin{bmatrix} -2 & 3 & 1 & 0 \\ 5 & -4 & 0 & 1 \end{bmatrix}$

and strive to find the equivalent matrix which ends up with:

$$\begin{bmatrix} 1 & 0 & \dfrac{A_{11}}{A} & \dfrac{A_{21}}{A} \\ 0 & 1 & \dfrac{A_{12}}{A} & \dfrac{A_{22}}{A} \end{bmatrix}$$

This is done via the following transformations.

$$\begin{bmatrix} -2 & 3 & 1 & 0 \\ 5 & -4 & 0 & 1 \end{bmatrix} \overset{-\frac{1}{2}R_1}{\sim} \begin{bmatrix} 1 & -3/2 & -\frac{1}{2} & 0 \\ 5 & -4 & 0 & 1 \end{bmatrix} \overset{-5R_1+R_2}{\sim} \begin{bmatrix} 1 & -3/2 & -\frac{1}{2} & 0 \\ 0 & 7/2 & 5/2 & 1 \end{bmatrix} \overset{\frac{2}{7}R_2}{\sim} \begin{bmatrix} 1 & -3/2 & -\frac{1}{2} & 0 \\ 0 & 1 & 5/7 & 2/7 \end{bmatrix}$$

$$\overset{\frac{3}{2}R_2+R_1}{\sim} \begin{bmatrix} 1 & 0 & 4/7 & 3/7 \\ 0 & 1 & 5/7 & 2/7 \end{bmatrix}$$

Therefore, the inverse is: $\begin{bmatrix} 4/7 & 3/7 \\ 5/7 & 2/7 \end{bmatrix} = \dfrac{\begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}}{7}$ as in example 12!

<u>EXAMPLE 4</u>: Find A^{-1} if $A = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ We have,

$$\begin{bmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix} \overset{-\frac{1}{3}R_1}{\sim} \begin{bmatrix} 1 & -2/3 & 1/3 & -1/3 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix} \overset{2R_1+R_2}{\sim}$$

$$\begin{bmatrix} 1 & -2/3 & 1/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & 8/3 & -2/3 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix} \overset{-3R_2}{\sim} \begin{bmatrix} 1 & -2/3 & 1/3 & -1/3 & 0 & 0 \\ 0 & 1 & -8 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix} \overset{-R_2+R_3}{\sim} \begin{bmatrix} 1 & -2/3 & 1/3 & -1/3 & 0 & 0 \\ 0 & 1 & -8 & 2 & 0 & -3 \\ 0 & 0 & 7 & -2 & 1 & 3 \end{bmatrix}$$

$$\overset{3R_1}{\sim} \begin{bmatrix} 3 & -2 & 1 & -1 & 0 & 0 \\ 0 & 1 & -8 & 2 & 0 & -3 \\ 0 & 0 & 7 & -2 & 1 & 3 \end{bmatrix} \overset{2R_2+R_1}{\sim} \begin{bmatrix} 3 & 0 & -15 & 3 & 0 & -6 \\ 0 & 1 & -8 & 2 & 0 & -3 \\ 0 & 0 & 7 & -2 & 1 & 3 \end{bmatrix} \overset{\frac{1}{3}R_1}{\sim} \begin{bmatrix} 1 & 0 & -5 & 1 & 0 & -2 \\ 0 & 1 & -8 & 2 & 0 & -3 \\ 0 & 0 & 7 & -2 & 1 & 3 \end{bmatrix}$$

$$\overset{\frac{1}{7}R_3}{\sim} \begin{bmatrix} 1 & 0 & -5 & 1 & 0 & -2 \\ 0 & 1 & -8 & 2 & 0 & -3 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{bmatrix} \overset{8R_3+R_2}{\sim} \begin{bmatrix} 1 & 0 & -5 & 1 & 0 & -2 \\ 0 & 1 & -0 & -2/7 & 8/7 & 3/7 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{bmatrix} \overset{5R_3+R_1}{\sim} \begin{bmatrix} 1 & 0 & -0 & -3/7 & 5/7 & 1/7 \\ 0 & 1 & 0 & -2/7 & 8/7 & 3/7 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{bmatrix}$$

Hence $A^{-1} = \dfrac{\begin{bmatrix} -3 & 5 & 1 \\ -2 & 8 & 3 \\ -2 & 1 & 3 \end{bmatrix}}{7}$ as in example 2.

<u>EXERCISES(17)</u>: If $A = \begin{bmatrix} 1 & 3 & -7 \\ 2 & 4 & 5 \\ -3 & 1 & 15 \end{bmatrix}$

1. Find (a) A_{13} (b) A_{23} (c) $|A|$ (d) A' (e) AA' (f) Adj A (g) What is peculiar about AA'? (h) $A'A$ (i) $A + A'$ (j) $|A + A'|$.

2. Find the inverse of $\begin{bmatrix} -3 & 4 \\ -10 & 2 \end{bmatrix}$ 3. Find the inverse of $\begin{bmatrix} 2 & -3 & 4 \\ 5 & 6 & -2 \\ -4 & 6 & -8 \end{bmatrix}$

4. Show that $\begin{bmatrix} 1 & 5 & -2 & 1 \\ 2 & -3 & 1 & 4 \\ -3 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 7 \\ 0 & 3 & -1 & 1 \end{bmatrix}$ by suitable row and column transformations.

5. If $A = \begin{bmatrix} -2 & 3 \\ -4 & 1 \end{bmatrix}$ Find adj(adj A).

6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (a) Find adj A (b) Find $|\text{adj } A|$ (c) What is adj A's rank? (d) Find A^{-1} (e) What is peculiar about adj A?

7. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$ Find (a) AB (b) $(AB)^{-1}$ (c) BA (d) $(BA)^{-1}$ (e) AA'

8. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ Find (a) adj A (b) $|A|$ (c) A^{-1} (d) (adj A)A (e) AA'

9. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ Find (a) adj A (b) A^{-1}

10. Find the inverse of: $\begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & -3 & 1 & -2 \\ 1 & 4 & -2 & 3 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

CHAPTER 5 - SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY USE OF MATRICES AND DETERMINANTS

38. EXPRESSION OF NON-HOMOGENEOUS LINEAR EQUATIONS IN MATRIX FORM

One of the main uses of matrices and determinants is the solution of systems of linear equations - specifically m equations in n unknowns. It will be recalled that these types of equations occured previously when discussing Linear Dependence and Independence. Later it will be seen that the knowledge of how to solve such systems will greatly expedite the solution of other geometric problems when they occur.

Essentially there are three cases where m linear equations in n unknowns are involved. Case 1 is where the number of equations exceed the number of unknowns i.e. where $m > n$. Case 2 is where the number of equations equals the number of unknowns, i.e. where $m = n$, and case 3 is where the number of unknowns exceed the number of equations, i.e. $m < n$. The theorems relevant to these cases are given below and the proofs of these theorems may be found in any good algebra textbook dealing with this subject matter.

First, consider m equations in n unknowns symbolized by:

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = k_1 \\ a_{12}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = k_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = k_m \end{matrix} \qquad \text{where } k_i \neq 0$$

(1) known as non-homogeneous equations when $k_i \neq 0$.

Second, consider these equations in matrix form, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} \qquad \text{where } k_i \neq 0. \qquad (2)$$

Next, we consider the augmented matrix of the system, defined as,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & k_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & k_m \end{bmatrix} \qquad (3)$$

WITH THESE EQUATIONS AND NOTATION IN MIND, WE CONSIDER THREE CONDITIONS.

Case 1: $m > n$

If the rank of the coefficient matrix in (2) is the same as the rank of the augmented matrix (3), then there is at least one solution(i.e. the equations are consistent). If however, the rank of the coefficient matrix is less than that of the augmented matrix, there is no solution. Of course, the rank of the coefficient matrix can never be greater than the rank of the augmented matrix in any case, since the coefficient matrix is always part of the augmented matrix.

EXAMPLE 1: Solve the system of equations: $\left.\begin{matrix} 2x + 3y = 1 \\ x - 2y = 4 \\ 4x - y = 9 \end{matrix}\right\}$ here, $\begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 4 & -1 \end{bmatrix}$ has rank 2,

and $\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ 4 & -1 & 9 \end{bmatrix}$ has rank 2 also since its 3×3 determinant equals zero. Hence, we have an unique solution, in this case. To find this solution, we may solve any two equations, say, $x - 2y = 4$, $8x - 2y = 18$, which gives $7x = 14$ and thus, $x = 2$ and $y = -1$. These values have to satisfy the 1st equation.

<u>EXAMPLE 2</u>: Solve the system:
$$\left.\begin{aligned} x - y &= 3 \\ 2x - y &= 4 \\ x + 3y &= -2 \end{aligned}\right\}$$
Here, $\begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}$ has rank 2,

BUT, $\begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 4 \\ 1 & 3 & -2 \end{bmatrix}$ has rank 3, since its 3 x 3 determinant's value = 3. Therefore, no solution exists!

<u>EXAMPLE 3</u>: Solve the system:
$$\left.\begin{aligned} x + y + z &= 1 \\ 2x - 2y + z &= 3 \\ 5x - 7y + 2z &= 8 \\ 7x + 11y + 8z &= 6 \end{aligned}\right\}$$
Now, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 5 & -7 & 2 \\ 7 & 11 & 8 \end{bmatrix}$ has rank 2,

AND, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 3 \\ 5 & -7 & 2 & 8 \\ 7 & 11 & 8 & 6 \end{bmatrix}$ also has rank 2, so that in this case there exist an infinity of solutions - namely,
$$x = 5 - 3k/4$$
$$y = -\frac{k+1}{4}$$
$$z = k$$

<u>Case 2</u>: m = n

If the determinant of the coefficient matrix is not zero then the equations are consistent and we have a unique solution.

<u>EXAMPLE 4</u>: Solve the system:
$$\left.\begin{aligned} 2x - y + z &= -1 \\ x - 2y + z &= -3 \\ 3x - y - z &= 4 \end{aligned}\right\} \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ 3 & -1 & -1 \end{vmatrix} \neq 0,$$
Therefore, the equations are consistent and x = 1, y = 1, z = -2.

If the determinant of the coefficient matrix is zero, then the equations are consistent if the rank of the coefficient matrix and the augmented matrix are equal, so that there will then exist at least 1 solution. If however, the rank of the coefficient matrix is less than that of the augmented matrix there is no solution and the equations are inconsistent. It should be noted that in the case where the ranks of the coefficient and augmented matrices are equal, it can be shown that exactly n - r of the unknown variables can be given arbitrary values. Hence, if we are dealing with 3 equations in 3 unknowns for example, we can never find a <u>unique</u> solution when the determinant of the coefficient matrix is zero, but we can always find solutions.

<u>EXAMPLE 5</u>: Solve the system: $\left.\begin{aligned} x + y + z &= 2 \\ 2x - 2y + z &= 4 \\ 7x + 11y + 8z &= 14 \end{aligned}\right\}$ Now, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 7 & 11 & 8 \end{bmatrix}$ has rank 2.

$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -2 & 1 & 4 \\ 7 & 11 & 8 & 14 \end{bmatrix}$ also has rank 2 since all 3 x 3 determinants = 0. There are 3 - 2 (i.e. $n - r$) variables that may be assigned arbitrary values.

Solving, we obtain $x = 3k + 2$, $y = k$, $z = -4k$(letting $y = k$). Thus, if $k = 1$, one solution would be $(5,1,-4)$.

<u>EXAMPLE 6</u>: Solve the system: $\begin{aligned} x + y + z &= 2 \\ 2x - 2y + z &= 4 \\ 7x + 11y + 8z &= -4 \end{aligned}$ Now, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 7 & 11 & 8 \end{bmatrix}$ has rank 2.

BUT, $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -2 & 1 & 4 \\ 7 & 11 & 8 & -4 \end{bmatrix}$ has rank 3 since $\begin{vmatrix} 1 & 1 & 2 \\ -2 & 1 & 4 \\ 11 & 8 & -4 \end{vmatrix} = -54 \neq 0.$

Therefore the equations are inconsistent and <u>have no solution</u>!

<u>Case 3</u>: $m < n$

In this case solutions exist only if the rank of the coefficient matrix is equal to the rank of the augmented matrix and $n - r$ variables may be assigned arbitrary values.

<u>EXAMPLE 7</u>: Solve the system: $\begin{aligned} x - 2y + z &= 3 \\ 2x - y + 2z &= 3 \end{aligned}$

The rank of $\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 2 \end{bmatrix} = 2$ and the rank of $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 3 \end{bmatrix} = 2.$

Therefore, solutions exist and letting $z = k$, we have, $x = 1 - k$, $y = -1$. A particular solution could be: $(1,-1,0)$.

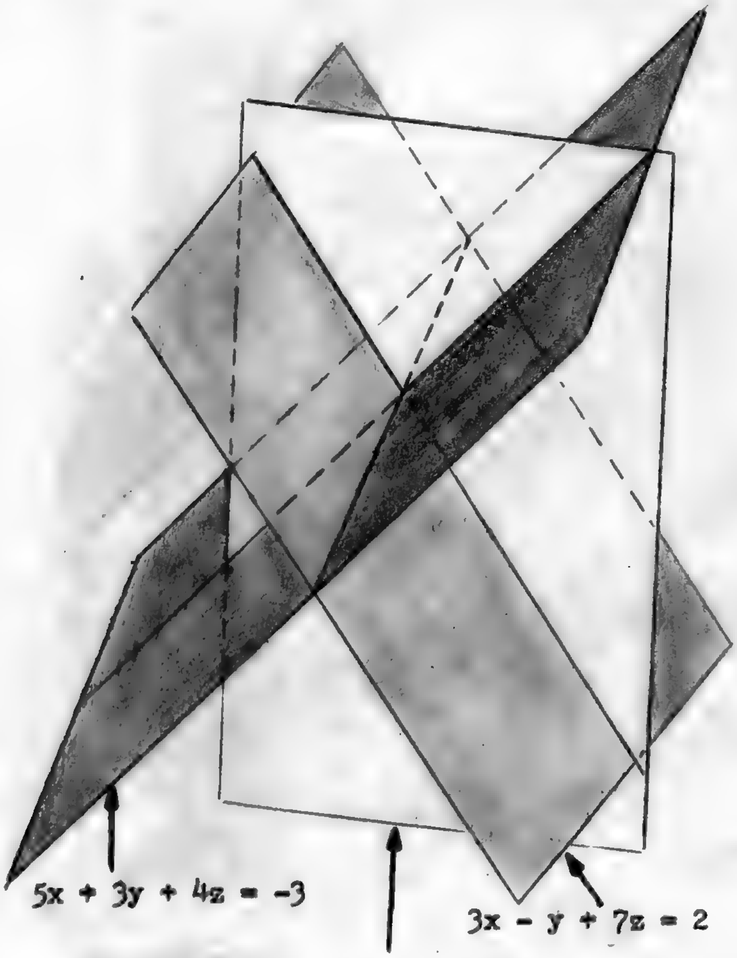
<u>EXAMPLE 8</u>: Solve the system: $\begin{aligned} x - 2y + z &= 3 \\ 2x - 4y - z &= 5 \end{aligned}$ Here, $r_c = r_a$ and the solution this time is $x = 8/3 + 2k$, $y = k$.

This is obtained by letting $z = t$, then get $2(3-t) = 5 + t$, so that $t = 1/3$. Then, $x - 2y = 3 - 1/3 = 8/3$; $2x - 4y = 5 + 1/3 = 16/3$. Then let $y = k$. Thus $x = 8/3 + 2k$.

<u>EXAMPLE 9</u>: Solve the system: $\begin{aligned} x - 2y + z &= 3 \\ 2x - 4y + 2z &= 5 \end{aligned}$ Here, $r_c < r_a$ and there is no solution!

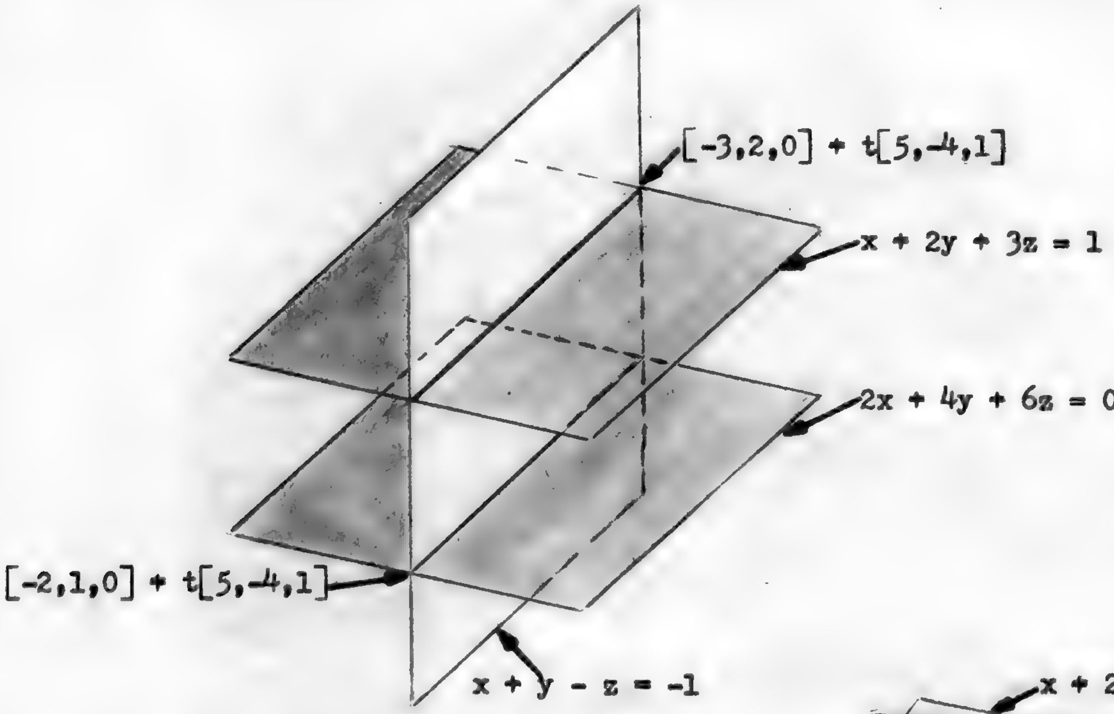
<u>EXAMPLE 10</u>: Solve the system of simultaneous linear equations:

$$\left.\begin{aligned} 3x - y + 7z &= 2 \\ x + y - 2z &= 1 \\ 5x + 3y + 4z &= -3 \end{aligned}\right\}$$

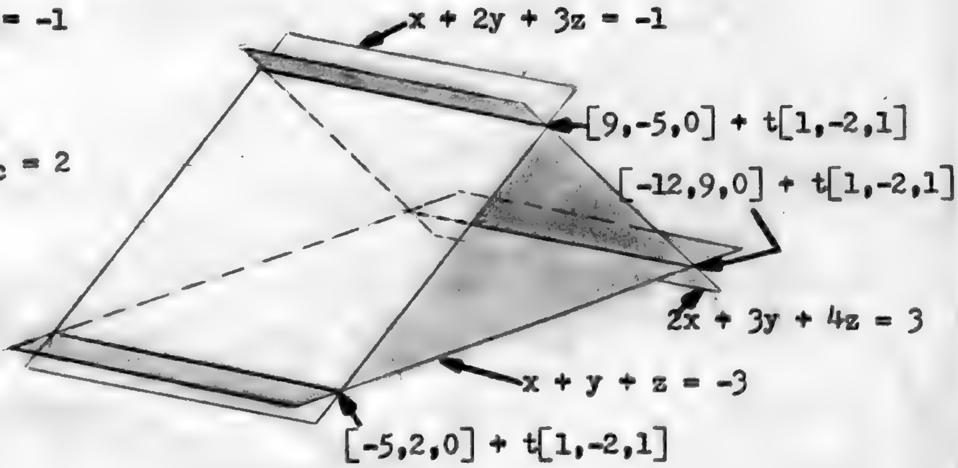


Here it is seen that $r_c = r_a = 3$ and hence there is a unique solution (2,-3,-1) to the system. Although we have not studied the geometric significance of 3 linear equations in 3 unknowns, we will assume for the moment that each equation represents a plane in space and the above system is amply illustrated on the left. In this case, of course, we immediately see that the planes will intersect in a common point - namely (2,-3,-1), the solution to the system above. Needless to say, there are other possiblities of planes intersecting besides the illustration above. Some of the more interesting cases are illustrated below and in each illustration, the equations of the planes and lines which are determined by one another are shown without explanation(except for the equations) since at this juncture, the student is not expected to realize the full ramifications of the illustrations presented. After the chapters dealing with planes and lines in 3-space, it would be advisable to the interested student to refer back to these drawings to appreciate the full geometric significance of them.

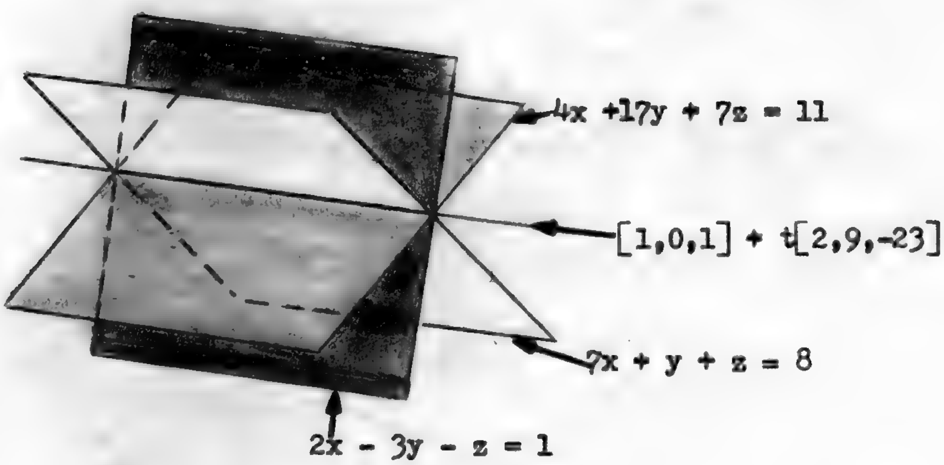
$r_a = 3,\ r_c = 2$



$r_a = 3,\ r_c = 2$



$r_a = r_c = 2$



To summarize then, we may have two types of sets of linear equations - the non-homogeneous and homogeneous types. The former is where at least one constant on the right hand side of the systems of equations is not zero and the following rules apply. First, the rank of the coefficient matrix can never be greater than that of the augmented matrix. Second, a system of linear equations is consistent if and only if the coefficient matrix has the same rank as the augmented matrix. Third, if the system of linear equations does have a solution which is not unique, and the coefficient and augmented matrices have rank r, then n - r of the unknowns may be assigned at pleasure(provided that the coefficient matrix of the remaining unknowns is also of rank r) and the others will then be uniquely determined.

39. HOMOGENEOUS LINEAR SYSTEMS IN MATRIX FORM

We must now deal with homogeneous systems of linear equations, i.e. where the constants on the right hand side of the equations are all zero. We have the matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

In this case it can readily be seen that the rank of the coefficient matrix is always equal to that of the augmented matrix(since the augmented matrix always contains a column of zeroes), and also, there is always a solution to the system, since the values $x_1 = x_2 = \cdots\cdots = x_n = 0$ will satisfy the equations. Such a solution is called the trivial solution.

Again we may have 3 cases, i.e. where $m > n$, $m = n$, $m < n$, but this time we are looking for a solution other than the trivial solution.

The theorems which apply to these systems are analgous to the ones that applied in the non-homogeneous case. First, a system of linear equations which are homogeneous has a solution other than the trivial solution if and only if the rank of the coefficient(and in this case the augmented, since they are the same) matrix is $< n$(the number of unknowns). Second, if the rank of the said system is r, the values of n - r of the unknowns may be assigned at pleasure and the others will be uniquely determined(provided that the coefficient matrix of the remaining unknowns is also

of rank r). To illustrate these concepts, we may subdivide into three cases m equations in n unknowns, i.e. $m > n$, $m = n$, $m < n$.

Case 1: $m > n$ (number of equations > number of unknowns).

EXAMPLE 1: Solve the system: $\left.\begin{aligned} 2x - y + 3z &= 0 \\ 3x + 2y + z &= 0 \\ x + 3y - 2z &= 0 \\ 5x + y + 4z &= 0 \end{aligned}\right\}$ Now the rank of $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix}$ is 2.

But $n = 3$ and since $r < n$, we must have a solution other than the trival solution. Let $z = k$ and solve any two of the equations for x and y say (1) and (4). We have,

$\left.\begin{aligned} 2x - y + 3k &= 0 \\ 5x + y + 4k &= 0 \end{aligned}\right\}$ $7x + 7k = 0$. Hence, $x = -k$, $y = k$. The general solution is $(-k,k,k)$.

EXAMPLE 2: Solve the system: $\left.\begin{aligned} x - y &= 0 \\ 3x + 2y &= 0 \\ 2x + y &= 0 \end{aligned}\right\}$ the rank of $\begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$ is 2 and $n = 2$, thus, only solution is $(0,0)$.

Case 2: $m = n$

EXAMPLE 3: Solve the system: $\left.\begin{aligned} 2x - 4y + 3z &= 0 \\ x + 2y - 2z &= 0 \\ x - 6y + 5z &= 0 \end{aligned}\right\}$ the rank of $\begin{bmatrix} 2 & -4 & 3 \\ 1 & 2 & -2 \\ 1 & -6 & 5 \end{bmatrix}$ is 2 and $n = 3$.

Since $r < n$, we must have another solution besides $(0,0,0)$; letting $z = k$, we have,

$\left.\begin{aligned} 2x - 4y + 3k &= 0 \\ 2x + 4y - 4k &= 0 \end{aligned}\right\}$ $4x - k = 0$. Hence, $x = k/4$, $y = 7k/8$. The general solution is: $(k/4,\ 7k/8,\ k)$.

EXAMPLE 4: Solve the system: $\left.\begin{aligned} 2x - 4y + 3z &= 0 \\ x + 2y - 2z &= 0 \\ x + y - 2z &= 0 \end{aligned}\right\}$ the rank of $\begin{bmatrix} 2 & -4 & 3 \\ 1 & 2 & -2 \\ 1 & 1 & -2 \end{bmatrix}$ is 3 since $\triangle = -7$

Since $n = r = 3$, there is <u>no solution</u>.

Case 3: $m < n$

EXAMPLE 5: Solve the system $\left.\begin{aligned} x - 2y + z &= 0 \\ x + 2y - 3z &= 0 \end{aligned}\right\}$ the rank of $\begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$ = 2, but $n = 3$.

Let $z = k$, we have, $\left.\begin{aligned} x - 2y + k &= 0 \\ x + 2y - 3k &= 0 \end{aligned}\right\}$ Hence $x = k$, $y = k$. Solution is (k,k,k).

Recapitulating very briefly by means of a block diagram, we have:

- **LINEAR EQUATIONS** m equations in n unknowns.
  - **HOMOGENEOUS** (constants always 0) always has trivial solution (0,0,0). Coefficient and augmented matrix ranks always equal.
    - $r = n$: no solution other than trivial.
    - $r < n$: solutions such that $n-r$ variables may be assigned arbitrary values.
  - **NON-HOMOGENEOUS** coefficient rank(r_c) always $\leq$ augmented matrix rank(r_a).
    - $r_c = r_a$: solutions such that $n-r$ variables can be assigned arbitrary values.
    - $r_c < r_a$: equations inconsistent. no solution.

<u>EXERCISES(18)</u>: Determine whether the following systems of equations are consistent or inconsistent, trivial solutions excluded.

1. $\left.\begin{aligned} 2x - y &= 4 \\ x - 3y &= 7 \\ 3x + 2y &= -1 \end{aligned}\right\}$

2. $\left.\begin{aligned} 3x + 2y - 3 &= 0 \\ 5x + 4y + 2 &= 0 \\ 3x + y &= 0 \end{aligned}\right\}$

3. $\left.\begin{aligned} 2x + 3y + z &= -1 \\ x + 5y - z &= -4 \\ 5x + 2y + 3z &= 3 \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x - 3y + 5z &= -5 \\ x - 3y + 4z &= -4 \\ x + 3y - 2z &= 2 \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x - y + z &= -5 \\ 3x + 2y - 2z &= 3 \end{aligned}\right\}$

6. $\left.\begin{aligned} 3x - 2y + z &= 1 \\ 2x + 4y - 5z &= 3 \\ 4x - 8y + 7z &= -2 \end{aligned}\right\}$

7. $\left.\begin{aligned} 3x + 2y - z &= 1 \\ 5x + 4y - 2z &= 0 \\ 2x - 3y - 5z &= 5 \\ x + y + z &= 1 \end{aligned}\right\}$

8. $\left.\begin{aligned} 2x + 3y - z &= 1 \\ x + 2y + z &= 3 \\ 5x + 8y - z &= 5 \\ x \qquad - 5z &= -7 \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x + 3y - z &= 1 \\ 5x + 2y + z &= -3 \\ x + 7y - 4z &= 5 \end{aligned}\right\}$

10. $\left.\begin{aligned} x + 7y + 2z + w &= 1 \\ x + y - z + 4w &= 0 \\ 2x + 2y - 2z + 5w &= 4 \\ 4x - 2y - 7z + 10w &= 11 \end{aligned}\right\}$

11. $\left.\begin{aligned} -2x + y + 2z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}\right\}$

12. $\left.\begin{aligned} x - 2y &= 0 \\ 2x - 4y &= 0 \end{aligned}\right\}$

13. $\left.\begin{aligned} x + 3y &= 0 \\ 2x - 3y &= 0 \end{aligned}\right\}$

14. $\left.\begin{aligned} x - y - 2z &= 0 \\ 2x + 5y - 3z &= 0 \\ 3x - 17y - 8z &= 0 \end{aligned}\right\}$

15. $\left.\begin{aligned} 2x - y + z &= 0 \\ 7x + 3y - z &= 0 \\ x + 5y + 2z &= 0 \end{aligned}\right\}$

16. $\left.\begin{aligned} x + 2y - 3z - w &= 0 \\ x - y + 2z - 2w &= 0 \\ x + 5y - 8z \qquad &= 0 \\ 2x - 5y + 9z - 5w &= 0 \end{aligned}\right\}$

17.
$$\begin{aligned} 2x + 3y + z - w &= 0 \\ x + y - z + 5w &= 0 \\ 3x + 5y + 3z - 7w &= 0 \\ x + 2y + 2z - 6w &= 0 \\ 5x + 6y - 2z + 14w &= 0 \end{aligned}$$

18.
$$\begin{aligned} 2x + y + 3z - w &= 0 \\ x - z - 4w &= 0 \\ 3x + 5y - z + 2w &= 0 \\ x + 2y - 3w &= 0 \\ 4x - y + 2z + w &= 0 \end{aligned}$$

19.
$$\begin{aligned} x + 5y - z + 2w &= 1 \\ 10x - 9y + 10z - 5w &= 0 \\ 2x - 3y + 3w &= -2 \\ 6x + 7y + 3z &= 3 \\ 7x - y + 4z + w &= 0 \\ 3x + 2y - z + 5w &= -1 \end{aligned}$$

20.
$$\begin{aligned} x + y - z + w &= 0 \\ 2x - 2y - 2z - 3w &= 0 \\ x + 3y - 2z - 3w &= 0 \\ 2x - 3y + z - 5w &= 0 \end{aligned}$$

40. METHODS OF SOLVING SIMULTANEOUS LINEAR EQUATIONS BY USING MATRICES AND DETERMINANTS

Aside from the usual algebraic and graphical methods involving substitution, elimination and the like, there are more expeditious methods using the concepts unfolded above. Essentially there are three methods which can be used with n equations in n unknowns(and here we will be concerned with square matrices) and one of these methods can also be used for m equations in n unknowns. The three methods are (1) Cramer's Rule - which uses the determinants of the coefficient matrix; (2) Inverse Matrix Method - which uses the inverse matrix to obtain a solution(if there is one); (3) Augmented Matrix Method - which uses the augmented matrix and can be used generally. We will scrutinize each of the methods below.

(1) CRAMER'S RULE: Suppose that we have a system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a & a & \cdots & a \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k \\ \vdots \\ k_n \end{bmatrix}$$

Then Cramer's Rule states that :

$$x_1 = \frac{\begin{vmatrix} k_1 & a_{12} & \cdots & a_{1n} \\ k_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ k_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{\triangle} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & k_1 & \cdots & a_{1n} \\ a_{12} & k_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & k_n & \cdots & a_{nn} \end{vmatrix}}{\triangle}$$

x_3 = etc., where $\triangle$ = determinant of coefficient matrix and $\neq 0$!

i.e. we replace the equation constants in the x_1 place, the x_2 place and so on, then divide by determinant value. If $\triangle = 0$, then we must resort to another method.

EXAMPLE 1: Solve the system: $\left.\begin{aligned} 2x - y + z &= -1 \\ x - 2y + z &= -3 \\ 3x - y - z &= 4 \end{aligned}\right\}$ now $\triangle = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 7.$

and $x = \dfrac{\begin{vmatrix} -1 & -1 & 1 \\ -3 & -2 & 1 \\ 4 & -1 & -1 \end{vmatrix}}{7} = \dfrac{7}{7} = 1$; $y = \dfrac{\begin{vmatrix} 2 & -1 & 1 \\ 1 & -3 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{7} = \dfrac{7}{7} = 1$; $z = \dfrac{\begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & -3 \\ 3 & -1 & 4 \end{vmatrix}}{7} = \dfrac{-14}{7} = -2$

Thus, the solution is (1,1,-2).

EXAMPLE 2: Solve the system: $\left.\begin{aligned} x + y + z &= 2 \\ 2x - 2y + z &= 4 \\ 7x + 11y + 8z &= 14 \end{aligned}\right\}$ Now $\triangle = 0$, therefore, we cannot use Cramer's rule!

(2) INVERSE MATRIX METHOD:

Suppose we have a system $Ax = k$ as above. Multiplying on the left by A^{-1}, we have $A^{-1}(Ax) = A^{-1}k$, but $A^{-1}(Ax) = (A^{-1}A)x = x$. Thus, our solution must be $x = A^{-1}k$, if $|A| \neq 0$, of course.

EXAMPLE 3: Solve the system: $\left.\begin{aligned} 2x - y + z &= -1 \\ x - 2y + z &= -3 \\ 3x - y - z &= 4 \end{aligned}\right\}$ Now $A^{-1} = \dfrac{\begin{bmatrix} 3 & -2 & 1 \\ 4 & -5 & -1 \\ 5 & -1 & -3 \end{bmatrix}}{7}$

Thus, $$\dfrac{\begin{bmatrix} 3 & -2 & 1 \\ 4 & -5 & -1 \\ 5 & -1 & -3 \end{bmatrix}}{7}\begin{bmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{\begin{bmatrix} 3 & -2 & 1 \\ 4 & -5 & -1 \\ 5 & -1 & -3 \end{bmatrix}}{7}\begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{\begin{bmatrix} 7 \\ 7 \\ -14 \end{bmatrix}}{7} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(3) AUGMENTED MATRIX METHOD:

This method uses the augmented matrix of the system and it does not matter how many equations or unknowns are involved or whether $\triangle = 0$. An equivalent matrix is found with as many zeroes as possible until it is convenient to determine the values of the unknowns.

EXAMPLE 4: Solve the system: $\left.\begin{aligned} 2x - y + z &= -1 \\ x - 2y + z &= -3 \\ 3x - y - z &= 4 \end{aligned}\right\}$ First, we form the augmented matrix: $\begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & -2 & 1 & -3 \\ 3 & -1 & -1 & 4 \end{bmatrix}$

Then we find an equivalent matrix which contains as many zeroes as possible.

$$\begin{bmatrix}2&-1&1&-1\\1&-2&1&-3\\3&-1&-1&4\end{bmatrix}\overset{R_1\leftrightarrow R_2}{\sim}\begin{bmatrix}1&-2&1&-3\\2&-1&1&-1\\3&-1&-1&4\end{bmatrix}\underset{-3R_1+R_3}{\overset{-2R_1+R_2}{\sim}}\begin{bmatrix}1&-2&1&-3\\0&3&-1&5\\0&5&-4&13\end{bmatrix}\overset{R_1+R_2}{\sim}\begin{bmatrix}1&1&0&2\\0&3&-1&5\\0&5&-4&13\end{bmatrix}\overset{-4R_2+R_3}{\sim}$$

$$\begin{bmatrix}1&1&0&2\\0&3&-1&5\\0&-7&0&-7\end{bmatrix}\overset{R_3\div 7}{\sim}\begin{bmatrix}1&1&0&2\\0&3&-1&5\\0&1&0&1\end{bmatrix}\underset{-3R_3+R_2}{\overset{-R_3+R_1}{\sim}}\begin{bmatrix}1&0&0&1\\0&0&-1&2\\0&1&0&1\end{bmatrix}$$

This immediately yields: $x = 1,\ y = 1,\ z = -2.$

<u>EXAMPLE 5</u>: Solve the system:

$$\left.\begin{aligned}x + y + z &= 2\\2x - 2y + z &= 4\\7x + 11y + 8z &= 14\end{aligned}\right\}$$

The augmented matrix is: $\begin{bmatrix}1&1&1&2\\2&-2&1&4\\7&11&8&14\end{bmatrix}$

$$\underset{-7R_1+R_3}{\overset{-2R_1+R_2}{\sim}}\begin{bmatrix}1&1&1&2\\0&-4&-1&0\\0&4&1&0\end{bmatrix}\overset{R_2+R_3}{\sim}\begin{bmatrix}1&1&1&2\\0&4&1&0\\0&0&0&0\end{bmatrix}\overset{-R_2+R_1}{\sim}\begin{bmatrix}1&-3&0&2\\0&4&1&0\\0&0&0&0\end{bmatrix}$$

This gives: $x - 3y = 2$

$4y = -z$

Let $z = -4k$, then $y = k$ and $x = 3k + 2$. Solution is $(3k + 2, k, -4k)$.

<u>EXAMPLE 6</u>: Solve the system:

$$\left.\begin{aligned}x - 2y + z &= 3\\2x - y + 2z &= 3\end{aligned}\right\}$$

The augmented matrix is: $\begin{bmatrix}1&-2&1&3\\2&-1&2&3\end{bmatrix}$

$$\overset{-2R_1+R_2}{\sim}\begin{bmatrix}1&-2&1&3\\0&3&0&-3\end{bmatrix}\overset{R_2\div 3}{\sim}\begin{bmatrix}1&-2&1&3\\0&1&0&1\end{bmatrix}\overset{2R_2+R_1}{\sim}\begin{bmatrix}1&0&1&1\\0&1&0&-1\end{bmatrix}$$

This gives $y = -1$

$x + z = 1.$

Let $z = k$, then $x = 1 - k$ and the solution is: $(1 - k, -1, k)$.

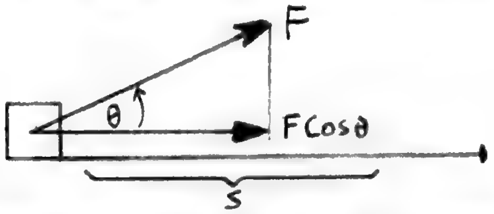
<u>EXERCISES(19)</u>: Solve the systems of linear equations in exercise 18 by using the three methods whenever possible.

<u>CHAPTER 6 - VECTOR MULTIPLICATION</u>

41. <u>DEFINITION OF VECTOR MULTIPLICATION</u>

It has been seen previously that it is possible to multiply a vector by a scalar to obtain another vector and geometrically it was seen that this new vector obtained was a vector which was considered either parallel or coincident but with a different magnitude. At the time of that discussion we deferred the concept of the product of two vectors. Now we shall deal with that concept and strive to adequately answer the question posed in the previous chapter. The first natural question to ask is how one defines the product of 2 or more vectors, and second, what will the answer represent - a scalar or a vector? Third, we might ask what the answer represents geometrically or physically.

Essentially, there are two answers, the product of two vectors can yield <u>either</u> a scalar or a vector. The scalar product is also known as the <u>dot product</u>; the vector product is referred to as the <u>cross-product</u>. The dot product of two vectors A and B is defined as $A \cdot B = |A||B|\cos\theta$ where θ is the angle between the vectors A and B. The motivation for such a definition came from physics - specifically, work-where to find the amount of work done by moving an object with a certain force F through a distance s, one had to find the component of the force in the direction of the distance. This, of course, would be given by $|F|\cos\theta$(see figure below) and the resulting work done would be $|F|s\cos\theta$. These and other considerations in geometry(such as projection) engendered the concept of the dot product. More will be said about projection in due course.



The cross-product between 2 vectors A and B is defined as $|A \times B| = |A||B|\sin\theta$ as one might expect. However, in this case, the cross-product yields a vector A x B and to be consistent, we have written the definition so that the left hand side is a scalar to conform with the right hand side which is also a scalar. Note that the magnitude of the vector A x B as defined above represents the area of a parallelogram determined by the two vectors A and B. Again, because of physical considerations, the vector is defined to be in the direction perpendicular to both A and B such that A, B and A x B form a right-handed system. Strictly speaking then, perhaps the definition should be in the form $\overrightarrow{A \times B} = \vec{N}|A||B|\sin\theta$ where $\vec{N}$ can be considered a unit vector perpendicular to both A and B so that A,B,N form a right-handed system. See figures 29 and 30.

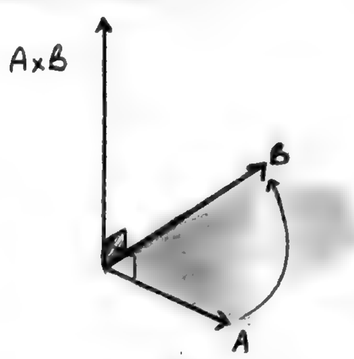


FIGURE 29

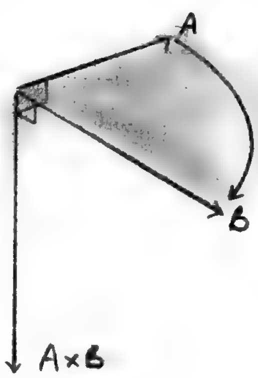


FIGURE 30

The easiest way to remember the direction of A x B is to think of tightening a screw where you are tightening from A to B as seen in figure 31 below:

FIGURE 31

Physical motivations for this definition came from mechanics(moments, m = r x F), electricity and magnetism(F = B⊥il) etc. where the direction of the forces agree with the above definition(see figures 32 and 33 below:).

There are several important things to notice. One is that the cross-product of A,B is peculiar to vectors A,B in 3-space only and that the cross-product yields a vector which is mutually perpendicular to both the vectors A and B. Another thing to note is that unlike the cross-product which is relevant for 3-space only, the dot product can be extended to a space of n dimensions as will be seen subsequently. Finally, we must remember that the dot product yields a scalar and the cross-product a vector. Note also that A x B = - (B x A) and therefore, A,B do not commute under the operation of vector multiplication.

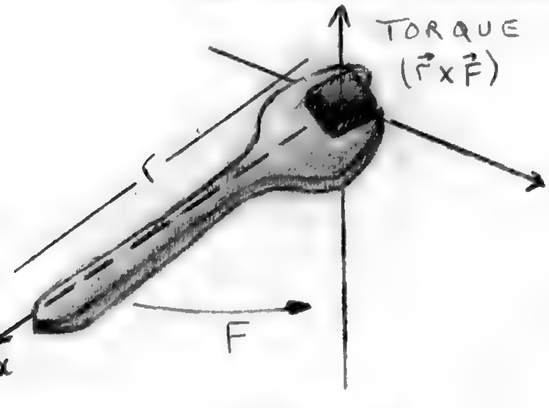


FIGURE 32

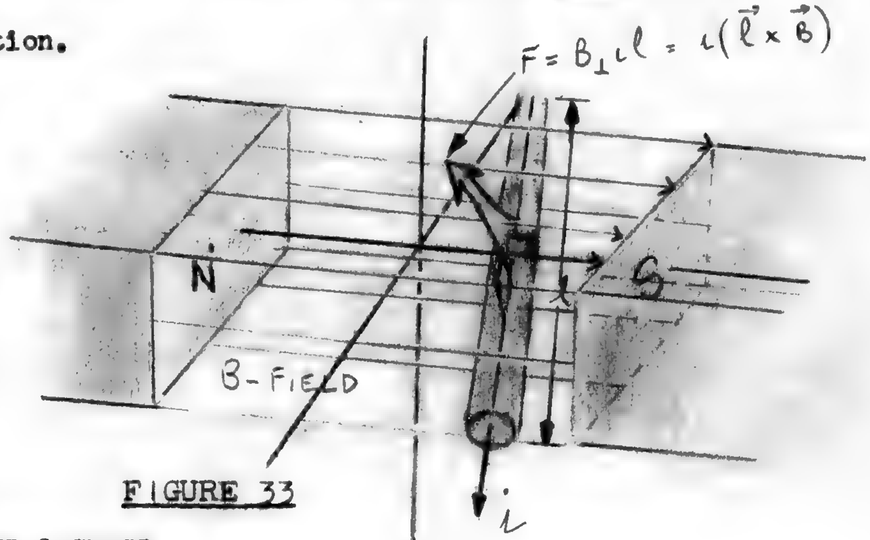
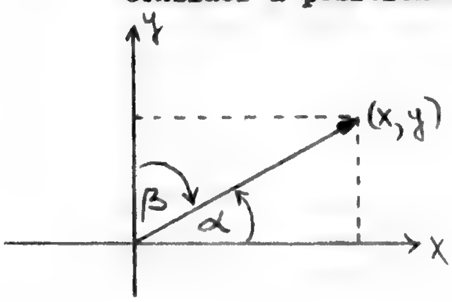


FIGURE 33

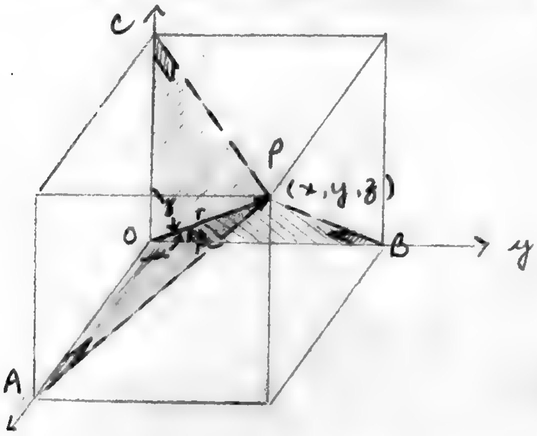
42. DIRECTION COSINES IN 2-SPACE AND 3-SPACE

Consider a position vector r in 2-space shown below:



The vector r forms angles α and β with the x and y-axes respectively. Now $\cos\alpha = x/|r|$ and $\cos\beta = y/|r|$. These cosines are called direction cosines in 2-space. Note that $\cos^2\alpha + \cos^2\beta = x^2/|r|^2 + y^2/|r|^2 = r^2/r^2 = 1$

Now consider a position vector r in 3-space as shown below:



Again, vector r forms angles α, β, γ with the x, y and z-axes respectively. In triangle OPA, $\cos\alpha = x/r$, in triangle OPB $\cos\beta = y/r$ and in triangle OPC, $\cos\gamma = z/r$. these values are called the direction cosines of the angles α, β, γ in 3-space. Again, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

We can, of course, extend this concept to any space or to any vector since vectors may be moved around at will so that if we have any vector whatsoever, we could "move" it so that it would become a position vector and hence we could define the direction cosines of such a vector in a similar manner in terms of its components, i.e. $\cos\alpha = A_1/|A|$; $\cos\beta = A_2/|A|$; $\cos\gamma = A_3/|A|$, where A is an arbitrary vector and A_1, A_2, A_3 are the components of the vector and are analgous to (x,y,z) above because of the vector's new position with respect to the origin.

<u>EXAMPLE 1</u>: Find the direction cosines of the position vector $[6, -6, 7]$.

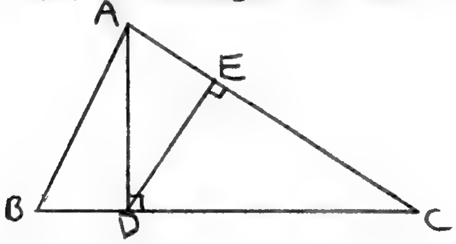
$\text{Cos}\,\alpha = \dfrac{6}{\sqrt{6^2 + 6^2 + 7^2}} = 6/11$; $\text{Cos}\,\beta = -6/11$; $\text{Cos}\,\gamma = 7/11$.

<u>EXAMPLE 2</u>: Find the direction cosines of the vector determined by (1,-1,14) & (6,-7,-16). The vector determined by these points is $[5,-6,-30]$. Therefore $\text{Cos}\,\alpha = 5/31$; $\text{Cos}\,\beta = -6/31$; $\text{Cos}\,\gamma = -30/31$.

<u>EXAMPLE 3</u>: Find the direction cosines of the vector determined by (1,2) and (-3,4). The vector determined by these points is $[-4,2]$ and thus $\text{Cos}\,\alpha = -4/\sqrt{20} = -2/\sqrt{5}$; $\cos\beta = 1/\sqrt{5}$.

43. <u>VECTOR AND SCALAR PROJECTION IN 2 AND 3-SPACE</u>

Recall that in elementary plane euclidean geometry that the projection of one line segment L_1 on another L_2 was determined by the perpendicular dropped form L_1 to L_2, i.e. in triangle ABC shown below, for example,

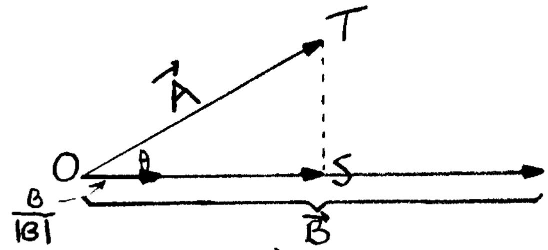


BD is the projection of AB on BC,

DE is the projection of AD on DE,

AE is the projection of AD on AC, etc.

Since we are dealing with straight lines, we may define the projection of one vector on another in any dimensional space and we do so in a manner similar to that of elementary geometry. Thus, if we have two vectors A,B the projection of A on B can either be a scalar quantity or a vector quantity, i.e. we may desire $\vec{OS}$ or $|\vec{OS}|$.

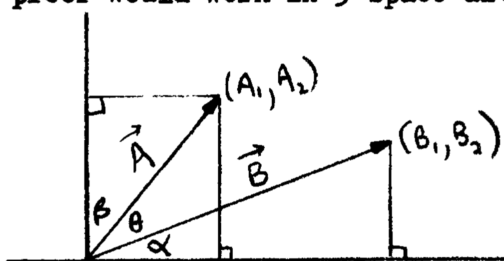


Now clearly, $|OS| = |A|\cos\theta$ and we call this the <u>scalar projection of A onto B</u>. To find the vector projection we simply express $\vec{B}$ in terms of a unit vector in the direction of $\vec{B}$. Now since $|OS| = |A|\cos\theta$ and since $|OS|\vec{B}/|B| = \vec{OS}$, then we have $\vec{OS} = |OS|\vec{B}/|B| = |A|\cos\theta\,\vec{B}/|B|$ and we call $\vec{OS}$ the <u>vector projection of A onto B</u>. Another way to express this is to multiply the right hand side of the equation by $|B|/|B|$ and we obtain $\vec{OS} = |A||B|\cos\theta\,\vec{B}/|B|^2$ or $(A\cdot B)\vec{B}/|B|^2$. Thus,

<u>Scalar proj A/B = $|A|\cos\theta$</u>; <u>Vector proj A/B = $(A\cdot B)\vec{B}/|B|^2 = |A|\cos\theta\,\vec{B}/|B|$</u>.

44. <u>A•B and A x B IN TERMS OF THEIR COMPONENTS</u>

We will deal with A•B first and derive a formula based on properties in 2-space only. By induction we can readily arrive at a general definition and a similar type of proof would work in 3-space although the derivation would be much more complicated.



As is seen in the figure, we have two position vectors A and B with their direction angles α, β and their components $[A_1, A_2]$ and $[B_1, B_2]$ as shown.

$\alpha + \beta + \theta = 90°$. Thus, $\theta = 90 - (\alpha + \beta)$. $\cos\theta = \cos[90 - (\alpha + \beta)] = \sin(\alpha + \beta)$. Therefore $\cos\theta = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, but $\sin\alpha = B_2/|B|$; $\cos\beta = A_2/|A|$; $\cos\alpha = B_1/|B|$; $\sin\beta = A_1/|A|$. Thus $\cos\theta = \dfrac{A_1B_1}{|A||B|} + \dfrac{A_2B_2}{|A||B|}$. Thus, $|A||B|\cos\theta = A_1B_1 + A_2B_2$.

Therefore, $A\cdot B = A_1B_1 + A_2B_2$ and by similar techniques in 3-space we can arrive at $A\cdot B = A_1B_1 + A_2B_2 + A_3B_3$ and in n-space $A\cdot B = A_1B_1 + A_2B_2 + \ldots\ldots A_nB_n$.

To derive an expression for A X B, we recall that the definition stated that A x B must be perpendicular to both A and B, i.e. we require a vector C(with components x,y,z, say) that is perpendicular to both A and B. From the definition of $A\cdot B = |A||B|\cos\theta$, we immediately see that if $\theta = 90°$, then $\cos\theta = 0$ and hence, A•B must

equal zero. This gives us a condition for perpendicularity. Thus our vector $C = [x,y,z]$ that we require, must satisfy two conditions: (1) $A \cdot C = 0$; (2) $B \cdot C = 0$. Therefore, we have $[A_1,A_2,A_3] \cdot [x,y,z] = 0$ and $[B_1,B_2,B_3] \cdot [x,y,z] = 0$, that is, $A_1x + A_2y + A_3z = 0$ and $B_1x + B_2y + B_3z = 0$. This has a solution if the rank of the matrix $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix}$ is less than 3(a non-trivial solution). Since this condition is met(by the nature of the matrix being a 2 x 3), we may solve the above system of equations but since we may have any component = 0, we may not use division in our attempt at a solution, since division by zero is not allowed. First, we eliminate z and we obtain:

$$
\begin{array}{l}
A_1B_3x + A_2B_3y + A_3B_3z = 0 \\
\underline{A_3B_1x + A_3B_2y + A_3B_3z = 0} \\
(A_1B_3 - A_3B_1)x + (A_2B_3 - A_3B_2)y = 0
\end{array}
$$

This will be satisfied if we put $A_2B_3 - A_3B_2 = x$ and $A_3B_1 - A_1B_3 = y$. Now plugging these values back in one of the origianl equations, we get,

$A_1(A_2B_3 - A_3B_2) + A_2(A_3B_1 - A_1B_3) + A_3z = 0$ which gives $A_2A_3B_1 - A_1A_3B_2 + A_3z = 0$.

$B_1(A_2B_3 - A_3B_2) + B_2(A_3B_1 - A_1B_3) + B_3z = 0$ " " $A_2B_1B_3 - A_1B_2B_3 + B_3z = 0$.

The solution for z is $A_1B_2 - A_2B_1$. Thus our required vector A x B has components: $[A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1]$. Fortunately, this may be put into determinant form for mnemonic purposes, i.e. $A \times B = \begin{vmatrix} i & A_1 & B_1 \\ j & A_2 & B_2 \\ k & A_3 & B_3 \end{vmatrix}$

<u>EXAMPLE 1</u>: If $A = [2,-3,6]$; $B = [-4,4,-7]$; Find (1) A x B (2) $A \cdot B$ (3) the angle between A and B (4) the scalar and vector projection of A on B (5) the direction cosines of A.

(1) $A \times B = \begin{vmatrix} i & 2 & -4 \\ j & -3 & 4 \\ k & 6 & -7 \end{vmatrix} = -3i - 10j - 4k = -[3,10,4]$

(2) $A \cdot B = [2,-3,6] \cdot [-4,4,-7] = 2(-4) + (-3)(4) + 6(-7) = -62$.

(3) $A \cdot B = |A||B|\cos\theta$ and hence, $\cos\theta = \dfrac{A \cdot B}{|A||B|} = \dfrac{-62}{7 \cdot 9} = -62/63$, $\theta = 169.8^{\circ}$ approximately.

(4) Scalar Proj$A/B = |A|\cos\theta = 7(-62/63) = -62/9$

Vector Proj $A/B = \dfrac{(A \cdot B)\ B}{|B|^2} = \dfrac{-62[-4,4,-7]}{81} = 62/81[4,-4,7]$ or $[248/81, -248/81, 434/81]$.

(5) $\cos\alpha = \frac{A_1}{|A|} = 2/7$; $\cos\beta = \frac{A_2}{|A|}$; $\cos\gamma = 6/7$.

__EXAMPLE 2__: Find a vector orthogonal(perpendicular) to $A[1,-1,3]$ and $B[3,-4,2]$.

The answer $A \times B$, Therefore we have, $\begin{vmatrix} i & 1 & 3 \\ j & -1 & -4 \\ k & 3 & 2 \end{vmatrix} = [10,7,-1]$. Note that any scalar t times this vector would be mutually orthogonal as well! To check: $[1,-1,3] \cdot t[10,7,-1] = 0$ & $[3,-4,2] \cdot t[10,7,-1] = 0$.

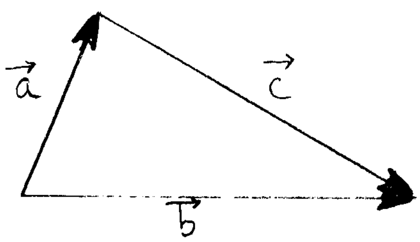
__EXAMPLE 3__: Show that $A\cdot(B + C) = A\cdot B + A\cdot C$.

If $A = [A_1, A_2, \ldots\ldots A_n]$; $B = [B_1, B_2, \ldots\ldots B_n]$; $C = [C_1, C_2, \ldots\ldots C_n]$;

Then $A\cdot(B + C) = [A_1, A_2, \ldots\ldots A_n]\cdot[B_1 + C_1, B_2 + C_2, \ldots\ldots B_n + C_n]$

$= A_1B_1 + A_1C_1 + A_2B_2 + A_2C_2 + \ldots\ldots + A_nB_n + A_nC_n = A\cdot B + A\cdot C$

__EXAMPLE 4__: Prove that in any triangle $|c|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$.



$c = b - a$, thus $|c| = |b - a|$

$|c|^2 = |b - a|^2$ but $|c|^2 = c\cdot c$ and

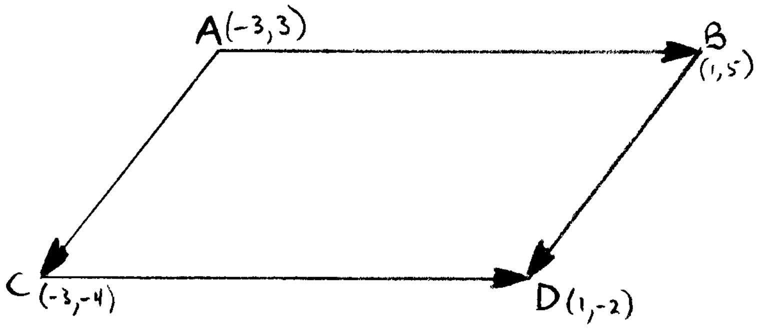
$|b - a|^2 = (b - a)\cdot(b - a)$. Therefore,

$c\cdot c = (b - a)\cdot(b - a) = b\cdot b - 2a\cdot b + a\cdot a$

Therefore, $|c|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$.

__EXAMPLE 5__: Find the area of the parallelogram $A(-3,3)$, $B(1,5)$, $C(-3,-4)$, $D(1,-2)$ in 2-space.

Ordinarily, we would use $|A \times B|$ in 3-space. We still can, but we make the third components of the vectors zero.



$\vec{AB} = [4,2] = \vec{CD}$

$\vec{AC} = [0,-7] = \vec{BD}$

$A \times B = \begin{bmatrix} i & 4 & 0 \\ j & 2 & -7 \\ k & 0 & 0 \end{bmatrix} = [0,0,28]$ and $|A \times B| = 28$ which is area!

__EXAMPLE 6__: Find the distance between $P_1(2,-6,-13)$ and $P_2(5,6,11)$.

Now $\vec{P_1P_2} = [3,12,24]$ and $|\vec{P_1P_2}| = 27 = d$.

__EXERCISES(20)__:

1. Given $P_1 = (-1,2,-2)$ and $P_2 = (3,10,17)$, find:
   (a) the distance between P_1 and P_2.
   (b) the direction cosine γ of $\vec{OP_1}$.
   (c) " " " α " $\vec{P_1P_2}$.

(d) the unit vector $\overrightarrow{OP_1}$.
(e) cosine of the angle between $\overrightarrow{OP_1}$ and $\overrightarrow{P_1P_2}$.

2. Given A = [2,-1,1], B = [1,2,-1], C = [1,1,-2] from the origin to P,Q,R respectively. Find:

(a) area of triangle OPQ.
(b) a unit vector perpendicular to plane OPQ.
(c) a vector perpendicular to plane OQR.
(d) a vector in plane OPQ perpendicular to C.
(e) area of parallelogram determined by B and C.

3. Given A = [1,-4,8], B = [-7,6,6], find:

(a) vector proj A/B (b) scalar proj A/B.

4. Given the vectors A = [1,-2,-3]; B = [2,1,-1]; find:

(a) A•B (b) $|A|$ (c) a vector perpendicular to A and B (d) the angle between the vectors
(e) area of parallelogram determined by A and B.

5. Find the distance between the following pairs of points:

(a) (3,-4) and (0,0) (b) (-5,-3) and (3,12) (c) (-1,3,-6) and (1,8,8) (d) (0,-3,8) and (8,6,-9) (e) (-2,2,5) and (2,6,-2).

6. Find unit vectors parallel to the vectors determined by the points:

(a) (11,-12) and (-10,16) (b) (10,12) and (-10,-9) (c) (0,1,-3) and (2,5,1) (d) (-1,-3,2) and (1,3,11) (e) (0,0,0) and (4,8,8).

7. Find the area of the parallelogram whose diagonals are determined by [3,1,-2] & [1,-3,4].

8. If the vectors A and B represent the sides of a rhombus, show by means of a scalar product that the diagonals are mutually perpendicular.

9. Prove the law of sines in a triangle by means of vector products.

10. By means of the scalar product show that every angle inscribed in a semicircle is a right angle.

11. Show that [2,3], [-3,2] and [5,1] form a right triangle.

12. Find the area of the triangle whose vertices are (0,1), (4,6) and (6,-1).

13. Given the vectors OA = [2,-1,3]; OB = [1,0,-1]; OC = [0,1,1]. Find the area of triangle ABC.

14. Find the area of the triangle whose vertices are (-1,2,3), (5,4,3) and (-2,-2,4).

15. If A = [-3,1,4]; B = [3,-1,5]; C = [-3,3,-4]; Find:

(a) A x (B x C) (b) (A x B) x C

45. TRIPLE PRODUCTS

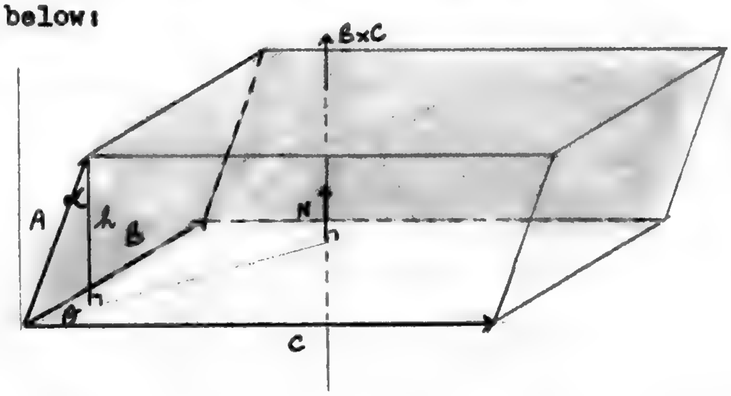
Recall that when the product of two vectors was explored it was found that there were two results - a scalar and a vector. The same is true for a product of 3 vectors. Hence, the triple product A•(B x C) will be a scalar and A x (B x C) will turn out to be a vector. First, let us see what A•(B x C) means both algebraically

and geometrically. Now $B \times C = \begin{vmatrix} i & B_1 & C_1 \\ j & B_2 & C_2 \\ k & B_3 & C_3 \end{vmatrix} = [B_2C_3 - B_3C_2, B_3C_1 - B_1C_3, B_1C_2 - B_2C_1]$.

Therefore, $A \cdot B \times C = [A_1, A_2, A_3] \cdot [B_2C_3 - B_3C_2, B_3C_1 - B_1C_3, B_1C_2 - B_2C_1]$

$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$, but this is exactly the same as $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Hence we may simplify the evaluation of the scalar triple product by use of a determinant. Note also that $A \cdot (B \times C) = C \cdot (A \times B) = B \cdot (A \times C) = (A \times B) \cdot C$ etc., since any of these expressions will represent a double switch of columns in the particular determinant and hence preserve the sign. To see what the scalar product means geometrically, consider the parallelpiped determined by three vectors A, B, C illustrated below:



Now $B \times C$ is a vector which is perpendicular to both B, C and the magnitude of $B \times C$ is equal to the area of the parallelogram determined by B and C, i.e. $|B \times C| = |B||C| \sin\theta$. Now the volume of the parallelpiped will be given by h times the area of the parallelogram determined by B and C; but $h = |A|\cos\alpha$. Multiplying both sides by $|\vec{N}|$ gives $h|N| = |A||N|\cos\alpha$ which implies that $h \cdot 1 = h = \vec{A} \cdot \vec{N}$ where $\vec{N}$ is a unit vector in the direction of h. We recall also that $B \times C = \vec{N}|B||C|\sin\theta$ where $\vec{N}$ is also in the direction of h and $|B||C|\sin\theta$ is the area of the parallelogram determined by B and C - call it a. Thus $B \times C = a\vec{N}$ and $h = \vec{A} \cdot \vec{N}$, but this says that $A \cdot (B \times C) = A \cdot (a\vec{N}) = a(A \cdot N) = ah =$ the volume of the parallelpiped determined by the vectors A, B and C.

Recall that $A \cdot B \times C$ was given by the determinant shown above so that if we had the expression $A \cdot (A \times C)$, two columns would be the same and the determinant's value would be zero. Thus the volume of the parallelpiped would be zero and the vectors would

have to be coplanar or collinear. We see immediately from the determinant theorems that $A\cdot(B \times C) = 0$ if and only if the vectors A, B, C are either coplanar or collinear, i.e. linearly dependent. Thus we have a rapid method of finding out whether 3 vectors are L.D. or not!

The algebraic representation of $A \times (B \times C)$ does not lead to anything serendipitous unfortunately, but there are formulas for conversion into simpler forms. The formula for such a conversion is the "BAC" minus "CAB" rule, i.e. $\underline{A \times (B \times C) = B(A\cdot C) - C(A\cdot B)}$. To establish this is quite simple but rather tedious, but to enlighten the student and familiarize the student with this particular type of proof, we herewith present the proof of the above rule. The left hand side becomes:

$$\begin{vmatrix} i & A_1 & B_2C_3 - B_3C_2 \\ j & A_2 & B_3C_1 - B_1C_3 \\ k & A_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

which equals: $[A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3,\ -A_1B_1C_2 + A_1B_2C_1 + A_3B_2C_3 - A_3B_3C_2,\ A_3B_2C_3 - A_3B_3C_2 + A_3B_1C_3 - A_3B_3C_1]$.

The right hand side becomes: $B(A_1C_1 + A_2C_2 + A_3C_3) - C(A_1B_1 + A_2B_2 + A_3B_3)$

$= [B_1, B_2, B_3](A_1C_1 + A_2C_2 + A_3C_3) - [C_1, C_2, C_3](A_1B_1 + A_2B_2 + A_3B_3)$

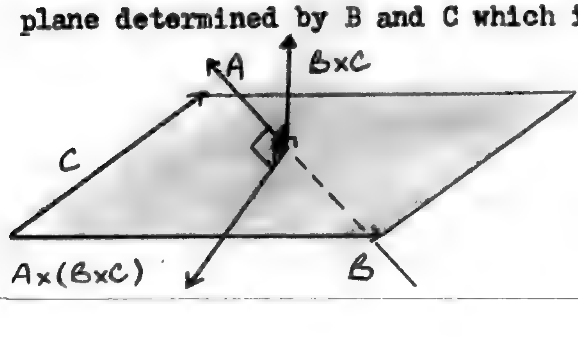
$= [\cancel{A_1B_1C_1} + A_2B_1C_2 + A_3B_1C_3 - \cancel{A_1B_1C_1} - A_2B_2C_1 - A_3B_3C_1,\ A_1B_2C_1 + \cancel{A_2B_2C_2} + A_3B_2C_3$
$- A_1B_1C_2 - \cancel{A_2B_2C_2} - A_3B_3C_2,\ A_3B_1C_3 + A_3B_2C_3 + \cancel{A_3B_3C_3} - A_3B_1C_3 - A_3B_2C_3 - \cancel{A_3B_3C_3}]$

which equals the left hand side.

To obtain $(A \times B) \times C$, we just recall that $(A \times B) \times C = -C \times (A \times B) = -[A(C\cdot B - B(C\cdot A)]$ by the "BAC" - "CAB" rule; but this last quantity $= B(C\cdot A) - A(C\cdot B)$ or $B(A\cdot C) - A(B\cdot C)$. This, of course, immediately shows us that the associative law does not hold for the cross-product of vectors!

Another useful formula is the <u>Lagrange Identity</u>. This says that for any vectors A, B, C, D that $(A \times B)\cdot(C \times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$ and represents a scalar. The proof can be established by a similar tedious process analgous to the proof of the above.

The geometric meaning of $A \times (B \times C)$ is shown below. It yields a vector in the plane determined by B and C which is perpendicular to A.



Note that the vector $A \times (B \times C)$ must be perpendicular to both A and $(B \times C)$ and hence parallel to both B and C. It must lie in the plane determined by B,C.

EXAMPLE 1: Find $(A + C)\cdot(C \times D)$.

The dot product distributes, hence, $(A+C)\cdot(C \times D) = A\cdot(C \times D) + C\cdot(C \times D) = A\cdot(C \times D)$, since $C\cdot(C \times D) = 0$.

EXAMPLE 2: Show that $A \times (B \times C) + B \times (C \times A) = (A \times B) \times C$.

$$
\begin{aligned}
A \times (B \times C) &= B(A\cdot C) - C(A\cdot B)\\
B \times (C \times A) &= C(B\cdot A) - A(B\cdot C)\\
\hline
A \times (B \times C) + B \times (C \times A) &= B(A\cdot C) - A(B\cdot C) = (A \times B) \times C. \text{ QED!}
\end{aligned}
$$

EXAMPLE 3: Given the points $A(3,-2,1)$; $B(-3,2-5)$; $C(-3,1,-2)$; find the volume of the pyramid OABC.

Since this is a four-point pyramid(i.e., half a prism) and since it takes 3 prisms to make up a parallelpiped, the volume will be given by 1/6 of the volume of a parallelpiped determined by the vectors A,B,C. Thus, $V = \dfrac{\begin{vmatrix} 3 & -3 & -3 \\ -2 & 2 & 1 \\ 1 & -5 & -2 \end{vmatrix}}{6} = 12/6 = 2.$

EXAMPLE 4: Show that $A = [1,2,-2]$; $B = [-1,4,2]$; $C = [4,-7-8]$ are L.D. and find scalars x,y,z such that $xA + yB + zC = 0$.

To show that A,B,C are L.D. we find $A\cdot(B \times C)$ which must equal zero.

Accordingly, $\begin{vmatrix} 1 & -1 & 4 \\ 2 & 4 & -7 \\ -2 & 2 & -8 \end{vmatrix} = 0$. Hence, A,B,C are L.D. and there must exist scalars such that $x[1,2,-2] + y[-1,4,2] + z[4,-7,-8] = 0$.

The coefficient matrix becomes: $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 4 & -7 \\ -2 & 2 & -8 \end{bmatrix} \underset{2R_1+R_3}{\overset{-2R_1+R_2}{\sim}} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 6 & -15 \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2\div 3}{\overset{2R_1}{\sim}} \begin{bmatrix} 2 & -2 & 8 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1+R_2}{\sim}$

$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ Therefore, we have $2x + 3z = 0$ which implies $x = -3z/2$

$2y - 5z = 0$ " " $y = 5z/2$.

Let $z = 2$, then an integral solution would be $x = -3$, $y = 5$.

Thus, $-3[1,2,-2] + 5[-1,4,2] + 2[4,-7,-8] = 0$ which checks!

EXAMPLE 5: Find $(A \times B) \times (C \times D)$.

Let $E = A \times B$, therefore, $E \times (C \times D) = C(E\cdot D) - D(E\cdot C) = C[(A \times B)\cdot D)] - D[(A \times B)\cdot C]$.

EXERCISES(21):

1. Show that $(A \times B)\cdot(C \times D) + (A \times D)\cdot(B \times C) = (A \times C)\cdot(B \times D)$.

2. Show that $[2,1-3]$; $[1,0,-4]$; $[4,3,-1]$ are L.D. and determine scalars x,y,z to verify this.

3. Given the vectors $A = [1,-4,8]$; $B = [-7,6,6]$; $C = [-7,4,-4]$; find:

(a) the volume ot the parallelpiped formed by these vectors.
(b) the area of triangle OAB.
(c) scalar and vector projection of B on C.
(d) a unit vector parallel to the plane determined by B and C and perpendicular to A.
(e) $(A \times B) \times C$.

4. Show that $A \times (A \times (A \times B)) = (B \times A)(A \cdot A)$.

5. Verify the following:

(a) $(A \times B) \times (A \times C) = A[A\cdot(B \times C)]$.

(b) $\begin{vmatrix} A\cdot A & A\cdot B \\ A\cdot B & B\cdot B \end{vmatrix} = |A \times B|^2$

(c) $A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$.

(d) $A \times (A \times (A \times (A \times (A \times B)))) = (A\cdot A)^2(A \times B)$.

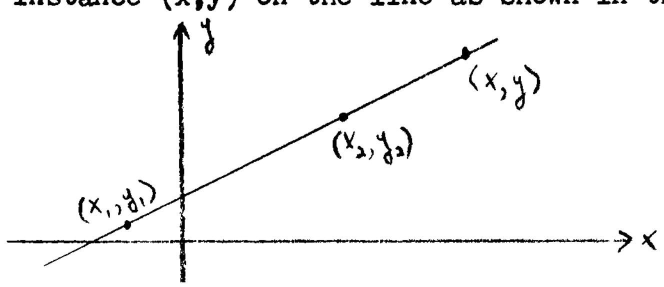
(e) $|A \times (A \times B)|^2 = |A|^4|B|^2 - |A|^2(A\cdot B)^2$.

CHAPTER 7 - THE STRAIGHT LINE IN 2-SPACE & 3-SPACE

46. THE STRAIGHT LINE IN 2-SPACE

The usual criterion that determines a straight line in 2-space is that two points determine a straight line. We will now use this postulate and our vector knowledge gleaned from the previous chapters to derive the two fundamental forms of the straight line, the cartesian form(the form involving x and y) and the parametric form(the form involving vectors directly and using some parameter t). Recall that a parameter can be considered as a "variable constant" - i.e. an oxymoron which means that we realize that the parameter is actually a certain number(and therefore not a variable, but a constant), but we can allow this number to assume different values depending on the circumstances.

Consider two points in two-space (x_1,y_1) and (x_2,y_2) and take a universal instance (x,y) on the line as shown in the figure below. A vector lying along the line would be given by $[x_2 - x_1, y_2 - y_1]$, $[x - x_1, y - y_1]$ or $[x - x_2, y - y_2]$ or any scalar multiple of these vectors. Thus, taking two of them containing x,y, we have,



$[x - x_1, y - y_1] = t[x_2 - x_1, y_2 - y_1]$, where t is any scalar(a parameter here!). Then,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = t.$$

This is called the cartesian form of the straight line determined by (x_1, y_1),(x_2,y_2). The value t, of course,

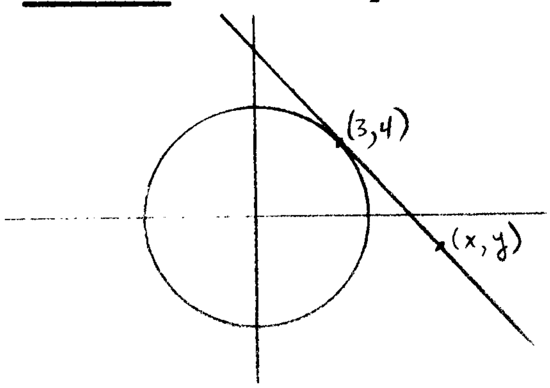
may be looked upon as a proportionality constant. Sometimes the COMPONENTS $x_2 - x_1$ and $y_2 - y_1$ are given special names. They are called the <u>direction numbers</u> of the line. They are symbolized by l, m in 2-space and l,m,n in 3-space. To obtain the <u>parametric form</u> of the line, we solve for $[x,y]$ in the above vector equations and we obtain either $[x,y] = [x_1,y_1] + t[x_2 - x_1, y_2 - y_1]$ or $[x,y] = [x_2,y_2] + t[x_2-x_1, y_2-y_1]$. The parametric form is very useful since it yields a point on the line and the vector lying along the line at once.

<u>EXAMPLE 1</u>: Find the cartesian and parametric equations of the line joining the points (1,-2) and (2,-3).

<u>Cartesian form</u>: $\frac{x - 1}{1} = \frac{y + 2}{-1} = t$ OR $\frac{x - 2}{1} = \frac{y + 3}{-1} = t.$

<u>Parametric form</u>: $[x,y] = [1,-2] + t[1,-1]$ <u>OR</u> $[2,-3] + t[1,-1]$.

<u>EXAMPLE 2</u>: Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3,4).

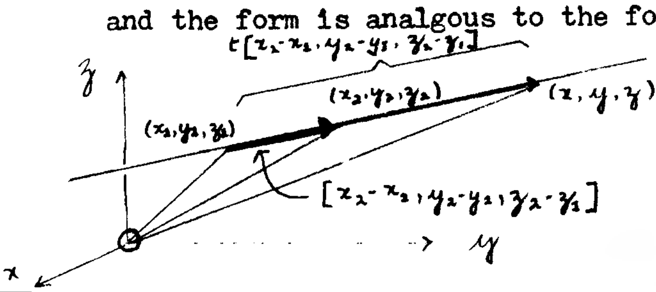


The position vector to the point (3,4) is $[3,4]$. Taking a universal instance (x,y) lying on the line, we obtain a vector $[x - 3, y - 4]$ lying along the line. Since the radius is perpendicular to the tangent at any point, the dot product must be zero. Accordingly,

$[x - 3, y - 4]\cdot[3,4] = 0$ which gives $3x - 9 + 4y - 16 = 0$, and thus, $3x + 4y - 25 = 0$.

47. <u>THE STRAIGHT LINE IN 3-SPACE</u>

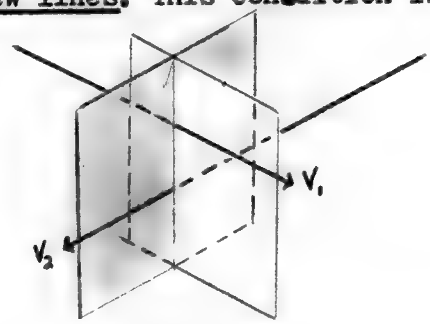
Again there are two forms of the straight line in 3-space - the cartesian and the parametric. However, there is more than one criterion that determines a straight line uniquely in 3-space; in fact there are two. One is that a line is uniquely determined by two points and the other is that a line is also uniquely determined by the intersection of two non-parallel planes. The former derivation we can do at once - the latter derivation we will defer until the next chapter which will deal with planes. The derivation of the equation of a line determined by 2 points is quite elementary and the form is analgous to the forms in 2-space. The triangle law is utilized.



Consider the position vectors $[x,y,z]$, $[x_1,y_1,z_1]$, $[x_2,y_2,z_2]$. Then a vector lying along the required line would be $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$ or any scalar multiple of this vector. Now

choose a scalar t such that it makes the length of the vector stretch(or shrink) to the universal instance (x,y,z) as shown in the figure. Then by the triangle rule, $[x_1,y_1,z_1] + t[x_2 - x_1,y_2 - y_1,z_2 - z_1] = [x,y,z]$, which is the parametric form of the line.. The cartesian form is: $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$ or another form:

$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = t$. Note that l,m,n are direction numbers and that the vector $[l,m,n]$ lies along the line. Two lines which are neither parallel nor meeting in 3-space are called <u>skew lines</u>. This condition is illustrated below:



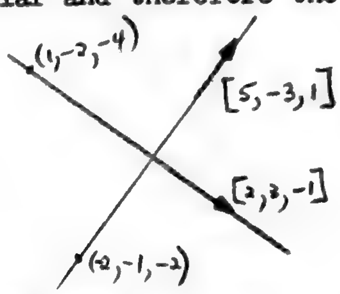
<u>EXAMPLE 1</u>: Find the parametric and cartesian forms of the lines determined by the points (1,-3,5) and (2,-2,3).

We immediately have: $\frac{x - 1}{1} = \frac{y + 3}{1} = \frac{z - 5}{-2} = t$ (cartesian form).

$[x,y,z] = [1,-3,5] + t[1,1,-2]$ (parametric form).

<u>EXAMPLE 2</u>: Show that the lines $\frac{x - 1}{2} = \frac{y + 2}{3} = \frac{z + 4}{-1}$ and $\frac{x + 2}{5} = \frac{y + 1}{-3} = \frac{z + 2}{1}$ are perpendicular assuming they are not skew.

If the lines are perpendicular, then the vectors lying along the lines must be perpendicular and therefore the dot product must be zero. Accordingly,



$t[2,3,-1]\cdot s[5,-3,1] = st(10 - 9 - 1) = 0$.

note that we may choose the vectors $[2,3,-1]$ and/or $[5,-3,1]$ omitting the scalars, since in this case, any suitable vectors lying along the line will suffice as our choice.

<u>EXAMPLE 3</u>: Show that the lines $\frac{x + 1}{2} = \frac{y - 1}{-1} = \frac{z + 3}{2}$ and $\frac{x + 1}{2} = \frac{y - 1}{3} = \frac{z + 3}{6}$ intersect and find the angle of intersection.

Since they share the common point (-1,1,-3), they must intersect and accordingly, we have $\cos\theta = \frac{A\cdot B}{|A||B|}$, where A,B are vectors lying along each line.

Therefore $\cos\theta = \frac{[2,-1,2] \cdot [2,3,6]}{3 \cdot 7} = 13/21$, and hence $\theta = 51.75°$.

EXAMPLE 4: Write the equations of the line through (2,-1,3) parallel to the x-axis.

A vector parallel to the x-axis must be of the form $t[1,0,0]$. Therefore the equation is:

$\frac{x-2}{1} = \frac{y+1}{0} = \frac{z-3}{0}$ or $y + 1 = 0$, $z - 3 = 0$.

EXAMPLE 5: Write the equations of the line through (2,-1,3) perpendicular to the lines whose direction numbers are 3,2,1 and 2,-2,3.

Since the direction numbers of L_1 and L_2 are 3,2,1 and 2,-2,3 respectively, a vector lying along L_1 would be [3,2,1] and along L_2 would be [2,-2,3]. Therefore a vector perpendicular to these two vectors would be given by the cross-product - namely: $[3,2,1] \times [2,-2,3] = [8,-7,-10]$; but [8,-7,-10] is a vector lying along the required line. Thus, the parametric equation would be $[2,-1,3] + t[8,-7,-10]$ and the cartesian equation would be: $\frac{x-2}{8} = \frac{y+1}{-7} = \frac{z-3}{-10}$.

EXAMPLE 6: Find the point of intersection(if any) of the lines: $\frac{x-1}{2} = \frac{y-5}{1} = \frac{z-4}{-7}$ and $\frac{x-14}{3} = \frac{y+2}{-3} = \frac{z-5}{5}$.

First, put the lines into parametric form, so that $L_1 = [1,5,4] + s[2,1,-7]$ and $L_2 = [14,-2,5 + t[3,-3,5]$. If there is a point (x,y,z) in common, then it must satisfy both equations simultaneously, i.e.,

$[x,y,z] = [1,5,4] + s[2,1,-7] = [14,-2,5] + t[3,-3,5]$. Hence,

$x = 2s + 1 = 3t + 14$; $y = s + 5 = -3t - 2$; $z = -7s + 4 = 5t + 5$. This gives us the system:

$$\begin{aligned} 2s - 3t &= 13 \\ s + 3t &= -7 \\ -7s - 5t &= 1 \end{aligned}$$

Now the rank of $\begin{bmatrix} 2 & -3 \\ 1 & 3 \\ -7 & -5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 13 \\ 1 & 3 & -7 \\ -7 & -5 & 1 \end{bmatrix} = 2$, since $\Delta = 0$.

Since $r_c = r_a$, there must be a solution and we use the augmented matrix method.

$$\begin{bmatrix} 2 & -3 & 13 \\ 1 & 3 & -7 \\ -7 & -5 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 3 & -7 \\ 2 & -3 & 13 \\ -7 & -5 & 1 \end{bmatrix} \underset{7R_1+R_3}{\overset{-2R_1+R_2}{\sim}} \begin{bmatrix} 1 & 3 & -7 \\ 0 & -9 & 27 \\ 0 & 16 & -48 \end{bmatrix} \underset{R_3 \div 16}{\overset{R_2 \div 9}{\sim}} \begin{bmatrix} 1 & 3 & -7 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \underset{R_2+R_3}{\overset{3R_2+R_1}{\sim}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $s = 2$, $t = -3$ and plugging these values back into the equations above for x,y,z, we obtain $x = 2(2) + 1$; $y = 2 + 5$; $z = -7(2) + 4$. The required point is (5,7,-10).

EXERCISES(22):

1. Find the cartesian and parametric equations of the following lines determined by the

indicated points:

(a) (-3,1), (-2,-3) (b) (5,-1), (5,-2) (c) (-1,3,4), (6,-2,3) (d) (4,-3,2), (0,0,0)
(e) (2,-3,1), (-2,-1,5).

2. Find the direction numbers and direction cosines of the above lines in problem 1.

3. Find the acute angles between 1(a) and 1(b) above.

4. Find the acute angles between 1(c) and 1(d) above; 1(c) and 1(e) above.

5. Write the equations of the line through (1,-2,3) parallel to the line through (-1,5,2) and (3,-4,3).

6. Write the equations of the line through (-1,5,2) and perpendicular to the lines:

$$\frac{x+1}{3} = \frac{y-2}{4} = \frac{z-2}{-1} \quad \text{and} \quad \frac{x-1}{2} = \frac{y+3}{-3} = \frac{z-2}{1}.$$

7. Show that the lines $\frac{x-7}{2} = \frac{y+9}{-2} = \frac{z-11}{3}$ and $\frac{x-11}{5} = \frac{y+5}{-1} = \frac{z+6}{-4}$ are non-skew and find their point of intersection. Verify that they are perpendicular.

8. Show that (2,-3,1), (5,4,-4) and (8,11,-9) are collinear.

9. Find the parametric form of the line $y + z = 0 = x - y$.

10. Find the equations of the line orthogonal to the pair of lines [1,2,16] + t[5,0,7] and [0,0,8] + t[-4,2,1] passing through their point of intersection.

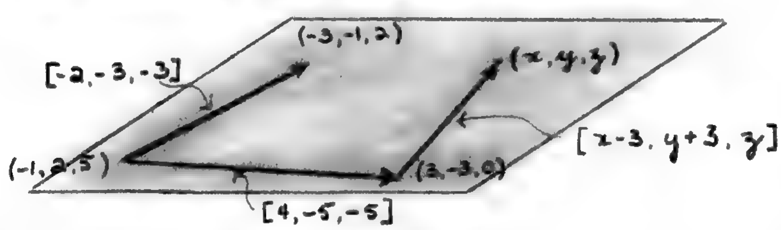
CHAPTER 8 - THE PLANE IN 3-SPACE

48. THE EQUATION OF THE PLANE

Just as there were 2 unique conditions for determining a straight line in space, there are 4 unique conditions for determining a plane in 3-space. (1) 3 non-collinear points (2) 2 non-skew intersecting lines (3) 2 parallel lines (4) a line and a point not on the line. Each of these types will be dealt with below by concrete examples.

<u>EXAMPLE 1</u>: Find the equation of the plane through the points (-1,2,5), (-3,-1,2) and (3,-3,0). (3 non-collinear points).

First, we illustrate the situation:

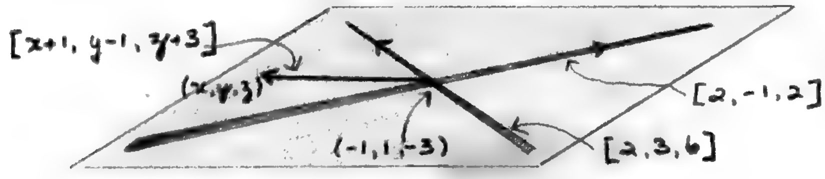


We then form vectors between the points as shown. Thus we have 2 vectors [-2,-3,-3] and [4,-5,-5]. Next, we take a universal instance (x,y,z) on the plane and form another vector with one of the points, say [x-3,y+3,z]. We know that these three vectors are coplanar and hence the triple product must be zero. Accordingly,

$$\begin{vmatrix} -2 & 4 & x-3 \\ -3 & -5 & y+3 \\ -3 & -5 & z \end{vmatrix} = 0. \text{ Thus } -2[-5z + 5(y+3)] + 3[4z + 5(x-3)] - 3[4(y+3) + 5(x-3)]$$

$$= y - z + 3 = 0.$$

EXAMPLE 2: Find the equation of the plane determined by the non-skew intersecting lines $\frac{x+1}{2} = \frac{y-1}{-1} = \frac{z+3}{2}$ and $\frac{x+1}{2} = \frac{y-1}{3} = \frac{z+3}{6}$. Again, we illustrate the situation:



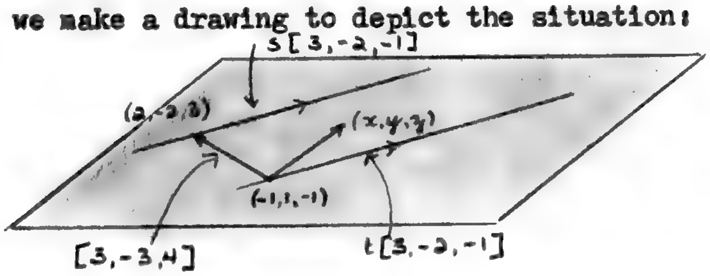
We recall that the direction numbers are the components of a vector lying along the line so that we have two vectors in the plane immediately. For our third vector, we merely take a universal instance (x,y,z) and form a vector with the point of intersection (-1,1,-3) which is common to both lines. Then we form the triple product as in example 1 and we have:

$$\begin{vmatrix} 2 & 2 & x+1 \\ -1 & 3 & y-1 \\ 2 & 6 & z+3 \end{vmatrix} = 2[3(z+3) - 6(y-1)] + 1[2(z+3) - 6(x+1)] + 2[2(y-1) - 3(x+1)],$$

which reduces to $3x + 2y - 2z - 5 = 0$.

EXAMPLE 3: Find the equation of the plane determined by the parallel lines: $\frac{x-2}{3} = \frac{y+2}{-2} = \frac{z-3}{-1}$ and $\frac{x+1}{-3} = \frac{y-1}{2} = \frac{z+1}{1}$.

Again, we make a drawing to depict the situation:

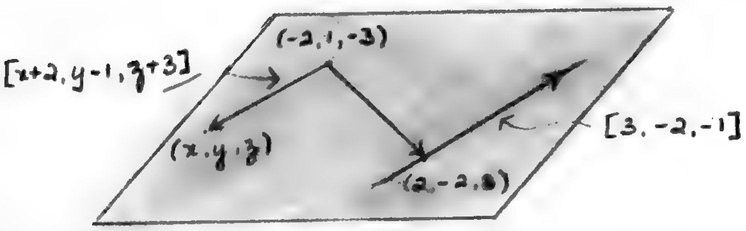


In this case we have 1 vector and 2 points, but we can find 2 other vectors - one from the two points given in the equations of the straight lines and one from the universal instance (x,y,z) and another point. Therefore we have,

$$\begin{vmatrix} 3 & 3 & x+1 \\ -2 & -3 & y-1 \\ -1 & 4 & z+1 \end{vmatrix} = 3[-3(z+1) - 4(y-1)] + 2[3(z+1) - 4(x+1)] - 1[3(y-1) + 3(x+1)],$$

which reduces to $11x + 15y + 3z - 1 = 0$.

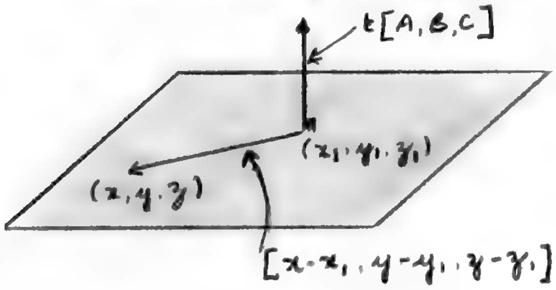
EXAMPLE 4: (a line and a point not on the line). Find the equation of the plane determined by the line $\frac{x-2}{3} = \frac{y+2}{-2} = \frac{z-3}{-1}$ and the point $(-2,1,-3)$.



Again we form three vectors and find the scalar triple product and we have:

$$\begin{vmatrix} x+2 & 4 & 3 \\ y-1 & -3 & -2 \\ z+3 & 6 & -1 \end{vmatrix} = (x+2)(15) - (y-1)(-22) + (z+3)(1) = 15x + 22y + z + 11 = 0.$$

From the above we seem to be safe in concluding that the general form of the equation of the plane in 3-space is $Ax + By + Cz + D = 0$ which indeed can be shown to be true. This, of course, is analgous to the linear form of the straight line in 2-space. We are now interested in a peculiar property of the equation when it is presented in this form. To see what this property is, let us suppose that we have the following situation:



We have a vector of some magnitude perpendicular to a plane through a point (x_1,y_1,z_1), and we wish to find the equation of the plane. (This could also be considered a way of defining a unique plane!).

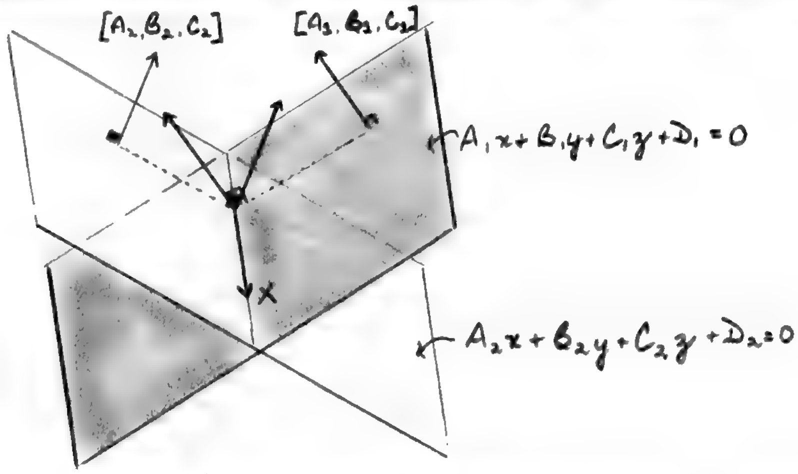
To do this, we immediately see that the dot product of the 2 vectors shown in the above drawing must be zero, i.e., $[x - x_1, y - y_1, z - z_1] \cdot t[A,B,C] = 0$. But this is the same as, $Ax + By + Cz - (Ax_1 + By_1 + Cz_1) = 0$, i.e. $Ax + By + Cz + D = 0$ where D is a constant. This is the general form of the equation of the plane in 3-space! Thus, our peculiar property becomes the fact that the coefficients of x,y and z form a vector which is always perpendicular to the plane!

EXAMPLE 5: Find a unit vector perpendicular to the plane $6x + 6y - 17z - 21 = 0$. A vector perpendicular to the plane must be $[6,6,-17]$ by the above discussion. Hence, a unit vector to the plane would be $\frac{[6,6,-17]}{19}$ or $[6/19, 6/19, -17/19]$. Another could be $[-6/19, -6/19, 17/19]$.

The above property gives us tremendous prowess to render otherwise difficult problems quite simple ofttimes.

49. PLANES AND LINES

Recall that in the previous chapter we deferred the discussion of the 2nd unique condition that determines a straight line-namely the intersection of 2 planes. We are now in a position to deal with this situation efficaciously. Consider the following figure:



We wish to determine how to derive the equation of the line of intersection of the two planes. If we consider for a moment that $[A_1, B_1, C_1]$ and $[A_2, B_2, C_2]$ are both normal vectors(perpendicular vectors) to each plane and we also consider that we would like to find a vector orthogonal to both these vectors(since this vector would lie along the line of intersection), we can see after a little reflection that this vector X has to be the cross-product of $[A_1, B_1, C_1]$ and $[A_2, B_2, C_2]$. Then to find the equation of the line, we may choose <u>any</u> point satisfying both equations of the plane and thus find the equation of the line.

<u>EXAMPLE 6</u>: Find the equation of the line determined by the planes $x + 2y - z - 3 = 0$ and $2x - 5y + 3z + 3 = 0$.

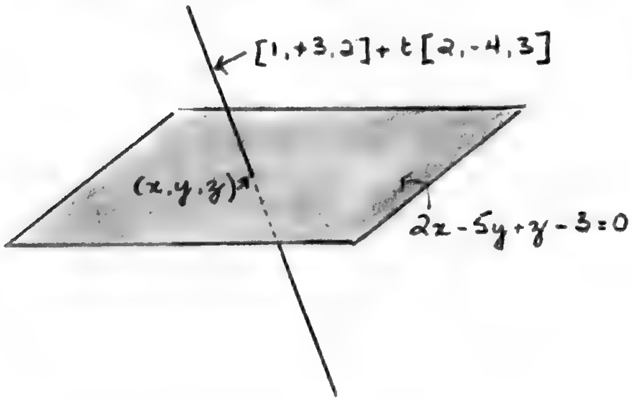
First, we find the orthogonal vectors to the plane. These are given by $[1,2,-1]$ and $[2,-5,3]$. Then we find the cross-product of these two vectors which gives us a vector lying along the line of intersection.

$$\begin{vmatrix} i & 1 & 2 \\ j & 2 & -5 \\ k & -1 & 3 \end{vmatrix} = [1,-5,-9]$$

Next, find any point satisfying both of the plane equations. This is done by letting $z = 0$, say. Then $x + 2y - 3 = 0$ and $2x - 5y + 3 = 0$. Hence, $x = 1$, $y = 1$ and therefore our straight line equation becomes $[1,1,0] + t[1,-5,-9]$ or in cartesian form: $\frac{x-1}{1} = \frac{y-1}{-5} = \frac{z}{-9}$.

There are a host of examples utilizing the properties of the plane and the line. One of the most frequent type is the following:

<u>EXAMPLE 7</u>: Find the point where the line $\frac{x - 1}{2} = \frac{y - 3}{-4} = \frac{z - 2}{3}$ intersects the plane $2x - 5y + z - 16 = 0$.



Now $x = 2t + 1$

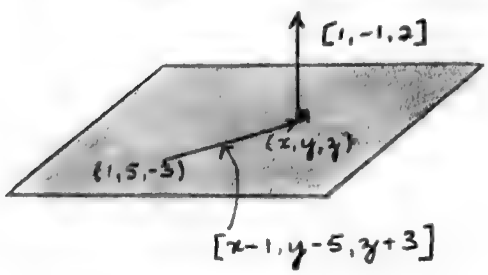
$y = -4t + 3$ from the line equation.

$z = 3t + 2$

and (x,y,z) must satisfy the equation of the plane. Hence, $2(2t + 1) -5(-4t + 3) + (3t + 2) - 16 = 0$. Thus, $4t + 2 + 20t - 15 + 3t + 2 - 16 = 0$. Therefore, $t = 1$, and we find that $(x,y,z) = (3,-1,5)$ from the relations above.

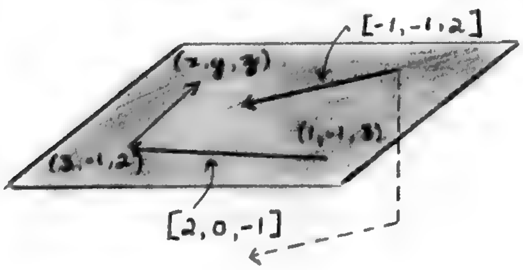
Other examples are shown below, but you will notice that they all utilize the basic intrinsic properties of the vector.

<u>EXAMPLE 8</u>: Find the equation of the plane through $(1,5,-3)$ orthogonal to $[1,-1,2]$.



Since $[1,-1,2]$ is perpendicular to the plane, it must therefore be perpendicular to <u>every</u> vector lying in the plane. Hence, $[x - 1,y - 5,z + 3] \cdot [1,-1,2] = 0$. Thus the equation is $x - y + 2z + 10 = 0$.

<u>EXAMPLE 9</u>: Find the equation of the plane containing the points $(1,-1,3)$, $(3,-1,2)$ and parallel to $[-1,-1,2]$.

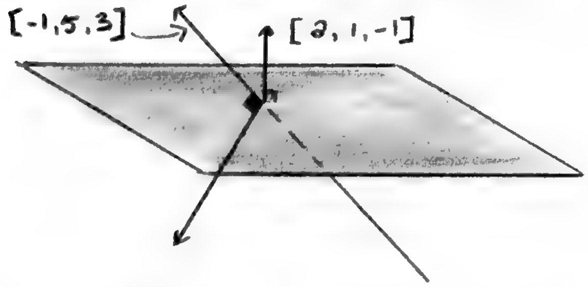


Since the plane is parallel to $[-1,-1,2]$, this is the same as saying $[-1,-1,2]$ is in the plane! Hence,
$$\begin{vmatrix} -1 & 2 & x - 3 \\ -1 & 0 & y + 1 \\ 2 & -1 & z - 2 \end{vmatrix} = 0.$$

Accordingly, $x + 3y + 2z - 4 = 0$.

<u>Another method</u>: Since the vector orthogonal to the plane has to be orthogonal to the vectors lying in the plane, then $[A,B,C] = [2,0,-1] \times [-1,-1,2] = [-1,-3,-2]$. Therefore, the equation would be: $-x - 3y - 2z + D = 0$. Now $(1,-1,3)$ has to satisfy this equation(since it lies in the plane) and so $-1 + 3 -2(3) + D = 0$. Thus, $D = 4$ and we have, $x + 3y + 2z - 4 = 0$ as found by method 1!

EXAMPLE 10: Find a non-zero vector parallel to the plane $2x + y - z + 3 = 0$ and orthogonal to $[-1,5,3]$.



The required vector must be simultaneously perpendicular to the vector perpendicular to the given plane and to $[-1,5,3]$. Hence it is equal to the cross-product of the vectors $[-1,5,3]$ and $[2,1,-1] = [-8,5,-11]$.

EXAMPLE 11: Find the cosine of the angle between the planes $7x - z + 1 = 0$ and $x + y - 1 = 0$.

The required angle will be the same as the angle between the normal vectors to the plane. Hence, $\cos\theta = \dfrac{[7,0,-1] \cdot [1,1,0]}{\sqrt{50}\sqrt{2}} = \dfrac{7}{10}$.

EXERCISES(23):

1. Find the equation of the planes through the following points:

(a) $(-3,2,-1)$, $(4,-2,1)$, $(3,-2,-3)$ (b) $(-2,1,3)$, $(0,0,0)$, $(1,-4,5)$ (c) $(4,-2,3)$, $(4,-1,1)$, $(4,3,2)$ (d) $(3,2,1)$, $(1,3,2)$, $(1,-2,3)$ (e) $(3,1,4)$, $(2,1,6)$, $(3,2,4)$.

2. Find the equation of the plane: (a) through the point $(1,-3,2)$ and parallel to the plane $3x - 2y + 5z - 3 = 0$. (b) through the point $(-2,3,4)$ and parallel to the plane $3x - 2z + 4 = 0$.

3. Find the equation of the plane:

(a) Through the point $(1,2,3)$ and perpendicular to the line $\dfrac{x-1}{3} = \dfrac{y+2}{-2} = \dfrac{z-2}{5}$

(b) " " " $(2,-1,4)$ & " " " " $y = 1$; $x = 0$.
(c) " " " $(-1,-2,-2)$ & " " each of the planes: $2x - y + 3z - 5 = 0 = 5x - 2y + 3z - 6$.
(d) " " " $(3,-2,1)$ & " " each of the planes: $2x + 3y - z + 4 = 0 = 4x + 6y - 2z - 7$.
(e) " " " " & parallel to each of the planes in (d) above.

4. Find the equation of the plane:

(a) through the points $(1,-2,-2)$, $(-3,1,-4)$ and perpendicular to $3x - 2y + z - 5 = 0$.
(b) " " " $(-1,2,5)$, $(3,2,-1)$ " " " $5x + 3y - 2z + 3 = 0$.

5. Find the acute angle between the following pairs of planes:

(a) $2x + 3y - z + 4 = 0 = x - y - z - 40$.
(b) $2x + 3y - z + 4 = 0 = 4x + 6y - 2z - 11$.
(c) $2x - y + z - 7 = 0 = x + y + 2z - 11$.
(d) $x + 2y - 2z + 5 = 0 = 6x - 5y + 30z - 4$.

6. Find the point or points of intersection (if any) of the planes:

(a) $2x - 3y + 5z - 6 = 4x - 3y + 2z + 4 = 3x + 2y - 3z + 5 = 0$.
(b) $3x - 2y - z - 4 = x - 2y - 2z + 3 = 0 = x - 2y + 2z - 4$.
(c) $2x - 3y + z - 5 = x - 3y + 4z - 6 = x + 3y - 10z + 8 = 0$.
(d) $2x - y + 3z + 4 = x + 2y + 5z - 6 = 7x - y + 14z - 3 = 0$.
(e) $2x - y + 3z + 4 = x + 2y + 5z - 6 = 3x - y + 4z - 2 = 0$.

7. (a) Find the equation of the plane passing through the intersection of the planes $x + 2y - z + 3 = 0 = 2x - y + z - 4$, and the point (1,2,3).

(b) Find the equation of the plane passing through the intersection of the planes $2x + 2y + 3z - 4 = 0 = 5x - 2y - 2z - 3$ and the point (3,-2,1).

8. Find the equation of the bisectors of the angles between the planes: $2x - y - 2z - 6 = 0 = 3x + 2y - 6z - 12$.

9. Find the point where the following lines pierce the following planes:

(a) $\frac{x-2}{3} = \frac{y+2}{4} = \frac{z-1}{-2}$; $5x - 2y + 3z - 16 = 0$.

(b) $\frac{x-2}{3} = \frac{y-1}{-2} = \frac{z-1}{1}$; $4x + 5y - 2z - 5 = 0$.

(c) $\frac{x+2}{1} = \frac{y-1}{1} = \frac{z-1}{4}$; $2x + 2y - z + 3 = 0$.

(d) $\frac{x+2}{2} = \frac{y-1}{2} = \frac{z-1}{-1}$; $4x - y + z - 7 = 0$.

10. Write the equations of the line in parametric and cartesian form whose equations are:

(a) $2x - 3y + 5z - 10 = 0 = 6x + 2y - 5z - 8$.
(b) $5x - 2y + z = 0 = 4x - 3y + 2z + 7$.
(c) $y + z = 0 = x - y$.
(d) $7x - z + 1 = 0 = x + y - 1$.

11. (a) Find the equations for the line through (1,2,3) orthogonal to the line, $\frac{x-1}{2} = \frac{y+1}{5} = \frac{z-2}{-3}$ and parallel to the plane $5x - 2y + 3z - 6 = 0$.

(b) Find the equations for the line through (-1,0,2) orthogonal to the line, $\frac{x+1}{-1} = \frac{y+2}{3}$; $z = 1$ and parallel to the plane $3x - 2y + 5z - 3 = 0$.

12. Find the equations for the line through (1,-2,-2) which is orthogonal to $x-3y+2z-10=0$.
" " " " " " " (-1,0,0) " " " " $2x-y+8z-5=0$.

13. (a) Find the equations of the line through (-2,3,-1) and perpendicular to the plane determined by the lines $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-3}{-1}$; $\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-3}{2}$.

(b) Find the equations of the line through (1,2,3) and perpendicular to the plane determined by the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{-1}$; $\frac{x-1}{-1} = \frac{y-2}{-2} = \frac{z-3}{2}$.
Where does this line intersect the plane?

(c) Find the equations of the line through (-1,0,-1) and perpendicular to the plane determined by the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+3}{-1}$; $\frac{x+1}{-1} = \frac{y-2}{-2} = \frac{z-3}{2}$.

(d) Find the equations of the line through (1,2,3) and perpendicular to the plane determined by the points (0,1,2), (-1,-2,1), (3,-2,1).

14. Find the acute angle between the following lines and planes:

(a) $\frac{x-5}{2} = \frac{y+3}{-1} = \frac{z-1}{-4}$; $3x + 5y - 6z - 7 = 0$.

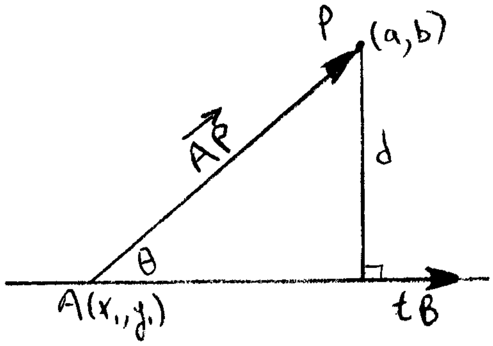
(b) $\frac{x-2}{3} = \frac{y-5}{2}$, $z = 1$; $5x - 3y - 2z - 8 = 0$.

15. (a) Find the equation of the line through (5,-2,3) orthogonal to the line $[1,2,3] + t[-1,-1,2]$ and parallel to the plane $x - z = 5$.

(b) At what point does the line $\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-5}{1}$ cut the sphere $(x-6)^2 + (y+4)^2 + z^2 = 49$?

50. DISTANCE FROM A POINT TO A LINE IN 2-SPACE AND 3-SPACE

When there is any ambiguity regarding the distance from one curve to another, unless otherwise specified, we always think of the perpendicular or shortest distance, i.e. that distance which will be a unique distance. In 2-space, of course, we mean the perpendicular distance from a point to a line. To obtain the required formula, we consider the following(which applies to 3-space as well!):



$\vec{B} = [1,m]$ in 2-space

$= [1,m,n]$ in 3-space

Now $\frac{d}{|AP|} = \sin\theta$. Thus, $d = |AP|\sin\theta$;

But $d\,|tB| = |tB|\,|AP|\sin\theta$, therefore,

$dt\,|B| = t\,|AP \times B|$. Hence $\boxed{d = \frac{|\vec{AP} \times \vec{B}|}{|B|}.}$

<u>EXAMPLE 1</u>: Find the distance from (1,2) to the line $2x - 3y + 6 = 0$.

The vector form of $2x - 3y + 6 = 0$ is $[-3,0] + t[3,2]$. Therefore, we have,

$\vec{AP} = (-3,0) - (1,2) = [-4,-2] = -2[2,1]$; $\vec{B} = [3,2]$. Thus,

$$d = \frac{|-2[2,1] \times [3,2]|}{\sqrt{13}} = \frac{\left| -2\begin{vmatrix} i & 2 & 3 \\ j & 1 & 2 \\ k & 0 & 0 \end{vmatrix} \right|}{\sqrt{13}} = \frac{-2[0,0,1]}{\sqrt{13}} = \frac{2}{\sqrt{13}}$$

Some mathematicians might consider it "bad form" to use the cross-product in 2-space but it's convenient and what's more important is that it works!

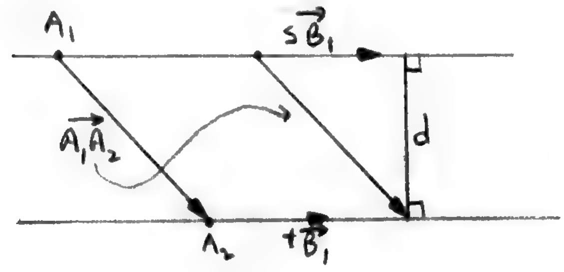
<u>EXAMPLE 2</u>: Find the distance from (1,2,3) to the line $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+3}{4}$.

$\vec{AP} = (-1,2,-3) - (1,2,3) = [-2,0,-6] = -2[1,0,3]$; $\vec{B} = [3,-2,4]$.

$$d = \frac{|-2[1,0,3] \times [3,-2,4]|}{\sqrt{29}} = \frac{|-2[6,5,-2]|}{\sqrt{29}} = \frac{2\sqrt{65}}{\sqrt{29}}$$ which approximately = 3.

51. DISTANCE BETWEEN TWO NON-SKEW PARALLEL LINES IN 2-SPACE OR 3-SPACE.

Using a similar technique to the derivation in 50.above, we may readily obtain the formula for the distance between 2 non-skew parallel lines $A_1 + sB_1$ and $A_2 + tB_1$. (Note that since the lines are parallel, they must have the same vector contained in them). Consider the figure below:



Here we take the vector $\overrightarrow{A_1A_2}$ and "move" it so that its terminal point(or initial point, as the case may be) coincides with the end of one perpendicular. Then we merely apply the formula above to obtain:

$$\boxed{d = \frac{|A_1A_2 \times B_1|}{|B_1|}}.$$

EXAMPLE 3: Find the distance between x - 2y + 4 = 0 and 2x - 4y - 5 = 0.

The parametric equation of L_1 is $[-4,0] + s[2,1]$, and of L_2 is $[5/2,0] + t[2,1]$.

$\overrightarrow{A_1A_2} = [13/2,0]$; $\vec{B_1} = [2,1]$. Thus, $d = \dfrac{|[13/2,0] \times [2,1]|}{\sqrt{5}} = \dfrac{6.5}{\sqrt{5}} = 2.9$ approximately.

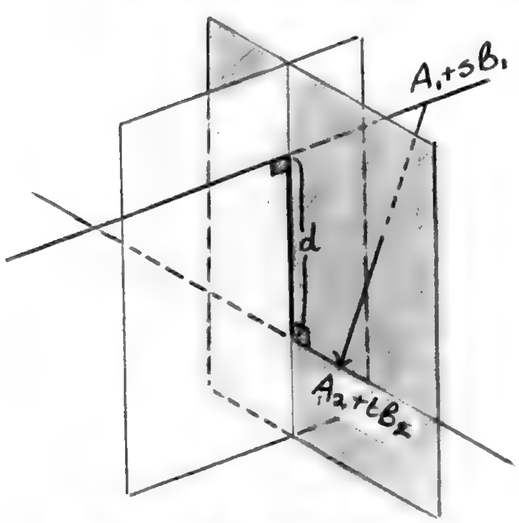
EXAMPLE 4: Find the distance between the lines: $\dfrac{x-1}{3} = \dfrac{y+2}{-5} = \dfrac{z-3}{-4}$ & $\dfrac{x+2}{-3} = \dfrac{y-3}{5} = \dfrac{z}{4}$

$\overrightarrow{A_1A_2} = (1,-2,3) - (-2,3,0) = [3,-5,3]$ and $\vec{B_1} = [3,-5,-4]$.

Therefore, $d = \dfrac{|[3,-5,3] \times [3,-5,-4]|}{|[3,-5,-4]|} = \dfrac{|[35,21,0]|}{|[3,-5,4]|} = \dfrac{7\sqrt{34}}{\sqrt{50}} = \dfrac{7\sqrt{2}\sqrt{17}}{\sqrt{5}\sqrt{2}} = 5.77$

52. DISTANCE BETWEEN SKEW LINES IN 3-SPACE

The distance that is required is, of course, the shortest distance between the skew lines and a little reflection will show that the vector lying along this distance must be mutually perpendicular to both lines. As a matter of fact, one can formally prove that such a perpendicular always exists and this immediately suggests a cross-product. The geometric situation looks something like this:



Since $\vec{d}$ is perpendicular to $\vec{B_1}$ and $\vec{B_2}$, then $\vec{d} = k(\vec{B_1} \times \vec{B_2})$ where k is some scalar. Now Project $\overrightarrow{A_1A_2}$ on $\vec{d}$ which will equal $\vec{d}$. We have,

$\dfrac{[\overrightarrow{A_1A_2} \cdot \vec{d}]\vec{d}}{|d|^2} = \vec{d}$. Thus, $[\overrightarrow{A_1A_2} \cdot \vec{d}] = |\vec{d}|^2$.

Substituting for $\vec{d}$ and $|\vec{d}|$, we have:

$k|\vec{B_1} \times \vec{B_2}||\vec{d}| = \overrightarrow{A_1A_2} \cdot k(\vec{B_1} \times \vec{B_2})$. Therefore,

$$\boxed{|\vec{d}| = \frac{|\overrightarrow{A_1A_2} \cdot (\vec{B_1} \times \vec{B_2})|}{|\vec{B_1} \times \vec{B_2}|}}$$ (distance always positive)

EXAMPLE 5: Find the shortest distance between the lines: $\dfrac{x-2}{3} = \dfrac{y+3}{4} = \dfrac{z-6}{-2}$ and

$\dfrac{x+1}{4} = \dfrac{y-3}{-2} = \dfrac{z-1}{1}$.

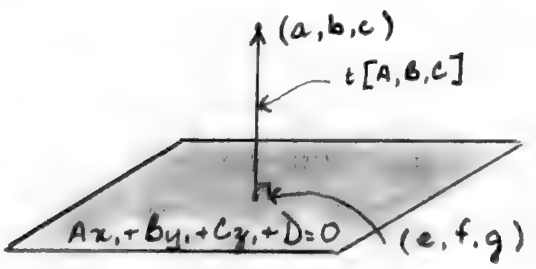
The parametric forms of the above lines L_1 and L_2 are:

$L_1 = [2,-3,6] + t[3,4,-2]$; $L_2 = [-1,3,1] + t[4,-2,1]$. Hence, $\overrightarrow{A_1A_2} = [3,-6,5]$ and $B_1 \times B_2 = [0,1,2]$. Therefore, $d = \dfrac{[3,-6,5] \cdot [0,1,2]}{\sqrt{5}} = \dfrac{4}{\sqrt{5}} = 1.78$ approximately.

Note that since any scalar "t" multiplied by $B_1 \times B_2$ will cancel out, we need not carry along large numbers and therefore we reduce the arithmetical tedium.

53. DISTANCE BETWEEN A POINT AND A PLANE IN 3-SPACE

The formula here derived is somewhat analgous to the classical plane analytic geometry formula giving the distance from a point to a line. The derivation is quite elementary. Consider the figure below:



We have $[a - e, b - f, c - g] = t[A,B,C]$ and $d = |t|\,|[A,B,C]|$. Thus, $d^2 = t^2(A^2 + B^2 + C^2)$, which gives $d = \sqrt{A^2 + B^2 + C^2}$ and therefore, $t = d/\sqrt{A^2 + B^2 + C^2}$. Now $e = a - tA$; $f = b - tB$; $g = c - tC$; but $Ae + Bf + Cg + D = 0$ and thus, $A(a - tA) + B(b - tB) + C(c - tC) + D = 0$, i.e., $Aa + Bb + Cc + D = t(A^2 + B^2 + C^2) = \dfrac{d(A^2 + B^2 + C^2)}{\sqrt{A^2 + B^2 + C^2}}$ and thus,

$$\boxed{d = \frac{Aa + Bb + Cc + D}{\sqrt{A^2 + B^2 + C^2}}.}$$

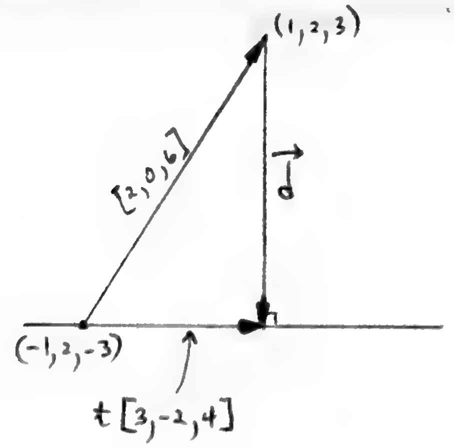
EXAMPLE 6: Find the distance from the point (1,2,3) to the plane $5x - 3y + 7z + 4 = 0$. Here, $(a,b,c) = (1,2,3)$ and a simple substitution in the formula yields:

$$\frac{5(1) - 3(2) + 7(3) + 4}{\sqrt{83}} = \frac{24}{\sqrt{83}} = 2.63 \text{ approximately.}$$

54. ANOTHER METHOD FOR DISTANCE DETERMINATION

Although the formulas above are convenient, one does not need to use them necessarily as long as the method of projection is familiar. For example, we can find the distance from a point to a line in 2-space or 3-space just as well by means of the projection formula. However, the distance formulas mitigate lengthy calculations and are the preferred method.

EXAMPLE 1: Find the distance from the point (1,2,3) to the line $\dfrac{x+1}{3} = \dfrac{y-2}{-2} = \dfrac{z+3}{4}$.



First, vector project [2,0,6] on t[3,-2,4] to obtain a value for t, i.e., proj [2,0,6]/t[3,-2,4]

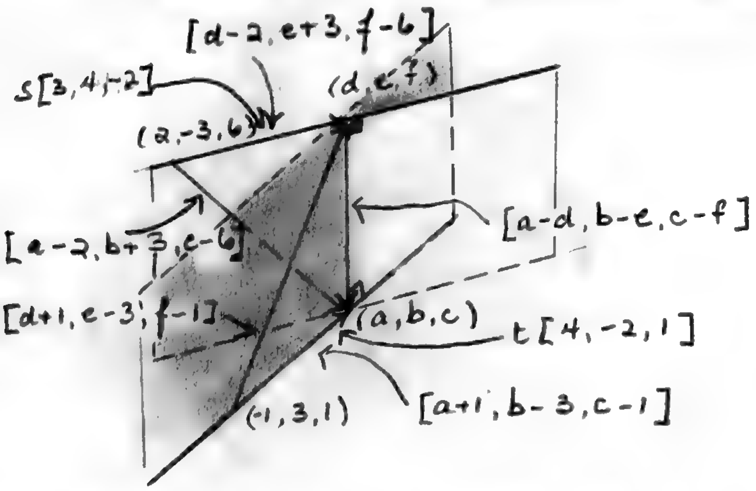
$$= \frac{([2,0,6] \cdot t[3,-2,4])t[3,-2,4]}{t^2 \cdot 29} = t[3,-2,4].$$

Cancelling and performing the indicated operations, we have, 30/29t = 1 and thus, t = 30/29.

Now $\vec{d}$ = 30/29[3,-2,4] - [2,0,6] = 1/29[32,-60,-54] = 2/29[16,-30,27]. Therefore $|d| = 2/29\sqrt{1885}$ $= 2\sqrt{65}/\sqrt{29}$ as in example 2, article 50.

EXAMPLE 2: Find the distance between the skew lines [2,-3,6] + t[3,4,-2] & [-1,3,1] +t[4,-2,1]

The method of solution is best illustrated by the following:



From the geometry of the situation, we have the following relations:

[d - 2,e + 3,f - 6] = s[3,4, -2],

[a + 1,b - 3,c - 1] = t[4,-2,1].

Thus, d = 3s + 2; e = 4s - 3; f = -2s + 6, and a = 4t - 1; b = -2t + 3; c = t + 1.

Now we project [a - 2, b + 3, c - 6] on s[3,4,-2] and [d + 1,e - 3,f-1] on t[4,-2,1] to obtain values for s and t. We therefore have:

[a - 2,b + 3,c - 6] = [4t - 3,-2t + 6,t - 5]; [d + 1,e - 3,f - 1] = [3s + 3,4s - 6,-2s + 5].

Projection 1: $\frac{[4t - 3,-2t + 6,t - 5] \cdot [3,4,-2]}{29s} = 1$. Thus, 29s = 2t + 25.

Projection 2: $\frac{[3s + 3,4s - 6,-2s + 5] \cdot [4,-2,1]}{21t} = 1$. Thus, 21t = 2s + 29.

Solving for s and t, we obtain, $s = .96\dot{3}\dot{6}$; $t = 1.4\dot{2}\dot{7}$. Now d = [a - d,b - e,c - f] = [4t - 3s - 3,-2t - 4s + 6,t + 2s - 5] = [0,-.8,-1.6] and d = 0^2 + .64 + 2.56 = $1.7\dot{8}$ as before in example 5, article 52.

Note that this method not only yields the distance but also the unique points on the lines (a,b,c) and (d,e,f); (a,b,c) = ($4.89\dot{0}\dot{9}$,$.05\dot{4}\dot{5}$,$2.47\dot{2}\dot{7}$) = (269/55,6/110,136/55).

(d,e,f) = ($4.89\dot{0}\dot{9}$,$.85\dot{4}\dot{5}$,$4.07\dot{2}\dot{7}$) = (269/55,47/55,224/55).

55. SUMMARY OF DISTANCE FORMULAS

Presented below are all of the distance formulas presented so far. The student

should be able to recall and use these formulas when necessary. They apply to both 2 and 3-space as shown.

	2-SPACE	3-SPACE
Distance between 2 points.	$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$	$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Distance between a point and a line.	$\frac{\lvert AP \times B\rvert}{\lvert B\rvert}$ or $\frac{\lvert Ax_1 + By_1 + C\rvert}{\sqrt{A^2 + B^2}}$	$\frac{\lvert AP \times B\rvert}{\lvert B\rvert}$
Distance between non-skew parallel lines.	$\frac{\lvert A_1A_2 \times B\rvert}{\lvert B\rvert}$	$\frac{\lvert A_1A_2 \times B\rvert}{\lvert B\rvert}$
Distance between skew lines.	----	$\frac{\lvert A_1A_2 \cdot B_1 \times B_2\rvert}{\lvert B_1 \times B_2\rvert}$
Distance between point (a,b,c) and plane Ax+By+Cz+D=0.	----	$\frac{\lvert Aa + Bb + Cc + D\rvert}{\sqrt{A^2 + B^2 + C^2}}$

EXERCISES(24):

1. Find the distance between the given points and the given lines: (a) $(-1,2,-3)$ and $\frac{x-2}{3} = \frac{y-1}{-4} = \frac{z-5}{-1}$; (b) $(2,-1)$ and $\frac{2x-1}{3} = \frac{y}{-2}$; (c) $(5,-3)$ and $\frac{x+2}{-2} = \frac{y+1}{3}$; (d) $(5,-2,-1)$ & $\frac{x-2}{3} = \frac{y-1}{2}$, $z = 1$; (e) $(-2,1,3)$ & $2x + 3y - 5z - 5 = 0$, and $3x - 2y + 5z - 1 = 0$.

2. Find the distance between the following lines: (a) $\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-1}{4}$ & $\frac{x-1}{-1} = \frac{y-1}{2} = \frac{z-1}{3}$; (b) $\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-1}{4}$ and $\frac{x+2}{-4} = \frac{y+3}{6} = \frac{z+3}{-8}$; (c) $\frac{x-1}{3} = \frac{y+2}{-4}$ and $\frac{x-2}{6} = \frac{y-2}{-8}$; (d) $\frac{x+1}{2} = \frac{y+1}{-2} = \frac{z-2}{2}$ and $[3,4,-1] + t[1,-1,1]$; (e) $\frac{x-1}{2} = \frac{y+1}{-2} = \frac{z-1}{3}$ and $\frac{x+1}{2} = \frac{y-1}{4} = \frac{z-5}{-3}$.

3. Find the distance between the points and planes listed: (a) $(-1,2,-3)$ and $x + y - 5z - 6 = 0$; (b) $(2,3,4)$ & $2x + 5y - 7z + 4 = 0$; (c) $(5,0,0)$ & $x = 2$; (d) origin & $5x - 6y + 7z + 8 = 0$; (e) $(5,4,-3)$ and $2x - 3y + 7z - 6 = 0$.

4. In problem 3 above, find the points that are closest to the given point in the plane.

5. (a) What is the equation of the sphere with center $(-4,-1,-2)$ tangent to the plane $2x - 2y + z + 26 = 0$?
   (b) What is the equation of the sphere with center $(2,5,-8)$ tangent to the plane $3x - 7y - 8z - 157 = 0$?

6. (a) Find the equation of the sphere whose center is $(1,2,3)$ and which passes through

the point (3,-4,-6).

(b) Find the equation of the plane through (3,-4,-6) which is tangent to the sphere in 6 (a) above.

7. Find the perimeter of the triangle determined by the vectors [8,1,-4], [-2,3,6] and [6,4,2].

8. Find the perpendicular distance between the planes:

(a) $2x + 5y - 14z + 20 = 0$ and $2x + 5y - 14z + 25 = 0$.
(b) $4x - 3y + 12z + 29 = 0$ and $4x - 3y + 12z - 10 = 0$.

9. Find the locus of a point whose distance from the plane $x + 2y - 2z = 12$ is always thrice the distance from $4x - y + 8z - 7 = 0$.

10. Determine whether the following lines are coplanar:

(a) $\frac{x-2}{-2} = \frac{y+1}{-3} = \frac{z-1}{4}$ and $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-3}{3}$.

(b) $\frac{x+3}{1} = \frac{y-3}{-1} = \frac{z+5}{-3}$ and $\frac{x+2}{1} = \frac{y-5}{1} = \frac{z+2}{1}$.

CHAPTER 10 - ORTHONORMAL BASES AND RECIPROCAL SETS OF VECTORS

56. ORTHONORMAL SETS OF VECTORS

Recall that n linear independent vectors spanned n-space and that if vectors were linearly independent, then 1 vector could always be expressed in terms of the others. Recall also that if there were n + 1 vectors in n-space they were always L.D. For example, 3 vectors would have to be L.D. in 2-space, 4 vectors would have to be L.D. in 3-space and so on. The simplest examples of this concept was our tacit use of the vectors i,j,k in 2 and 3-space and the fact that we can always express any vector in terms of these special unit vectors along the axes.

The question naturally arises now as to whether we can use any set of L.I. vectors as a means of expressing any other vector in a particular space. The answer is that we can-as long as the vectors we define are L.I. Thus, we need not have the vectors orthogonal or the magnitudes unity! This means that we need not have vectors expressed in terms of i,j,k and further, we can have an oblique system of coordinates which might prove useful in certain instances but whose use would make the dot and cross-products very unwieldy due to the introduction of cross-product terms that normally become zero when the coordinate system is rectangular.

We have seen how to express a vector D in 3-space as a linear combination of A, B,C, for example, by the use of the techniques developed in the chapter on the solution of simultaneous equations(linear equations). There is another formula which also

can be derived to yield the same result-i.e., to express D in terms of any three L.I. vectors A,B,C. The formula is:

$$D = \frac{(B\cdot C \times D)A}{(A\cdot B \times C)} - \frac{(A\cdot C \times D)B}{(A\cdot B \times C)} + \frac{(A\cdot B \times D)C}{(A\cdot B \times C)}$$

<u>EXAMPLE 1</u>: Show that A[1,-1,2]; B[3,0,-1]; C[2,-2,3] are L.I. and express D[-5,4,2] as a linear combination of them.

<u>Method 1</u>: First we establish that A,B,C are L.I. Now, $\begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -1 \\ 2 & -2 & 3 \end{vmatrix} = -3 \neq 0$. Hence the vectors are L.I.

The coefficient matrix is: $\begin{bmatrix} 1 & 3 & 2 & -5 \\ -1 & 0 & -2 & 4 \\ 2 & -1 & 3 & 2 \end{bmatrix} \underset{-2R_1+R_3}{\overset{R_1+R_2}{\sim}} \begin{bmatrix} 1 & 3 & 2 & -5 \\ 0 & 3 & 0 & -1 \\ 0 & -7 & -1 & 12 \end{bmatrix} \underset{7/3 R_2+R_3}{\overset{-R_2+R_1}{\sim}} \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & -1 & 29/3 \end{bmatrix} \overset{2R_3+R_1}{\sim}$

$\begin{bmatrix} 1 & 0 & 0 & 46/3 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & -1 & 29/3 \end{bmatrix}$ Therefore we have, $x + 46/3w = 0$; $3y - w = 0$; $-z + 29w/3 = 0$

Let $w = -1$, then $x = 46/3$, $y = -1/3$, $z = -29/3$

<u>Method 2</u>: The coefficient of A is $\frac{(B\cdot C \times D)}{-3} = \frac{\begin{vmatrix} 3 & 2 & -5 \\ 0 & -2 & 4 \\ -1 & 3 & 2 \end{vmatrix}}{-3} = -46/-3 = 46/3.$

The coefficient of B is $\frac{-A\cdot C \times D}{-3} = -\frac{\begin{vmatrix} 1 & 2 & -5 \\ -1 & -2 & 4 \\ 2 & 3 & 2 \end{vmatrix}}{-3} = -(-1/-3) = -1/3$

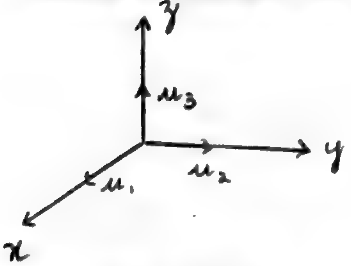
The coefficient of C is $\frac{A\cdot B \times D}{-3} = \frac{\begin{vmatrix} 1 & 3 & -5 \\ -1 & 0 & 4 \\ 2 & -1 & 2 \end{vmatrix}}{-3} = 29/-3 = -29/3$

These values agree with the above: Therefore, $D = 46/3\ A - 1/3\ B - 29/3\ C$.

Most of the time we are interested in mutually perpendicular sets of vectors whose magnitudes are unity. These types of L.I. sets of vectors are called <u>orthonormal sets</u> and it can be shown that if we adopt a right-handed coordinate system, the dot and cross-products will not change form when we change coordinate systems even though we may use different orthonormal sets; i.e., vectors are <u>invariant</u> under translation and rotation and the dot and cross-products are preserved as well.

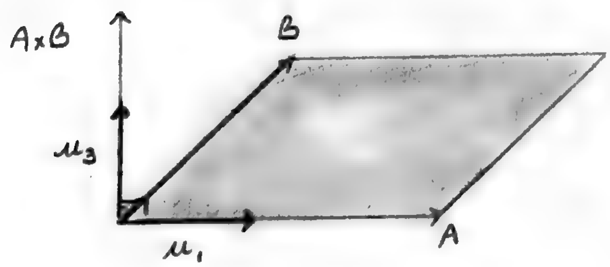
57. O-N SETS FOR ANY TWO VECTORS A, B IN 3-SPACE

Recall that by a right-handed system, we mean a consistently counter-clockwise system starting with the 1st coordinate axis as illustrated:



We would like our O-N set to be oriented in this fashion.

The easiest way to proceed is to take the first vector A and normalize it - i.e. make a unit vector out of it. Therefor $u_1 = A/|A|$. Thus, we might have a situation that looks like that below where the vectors are oriented as shown:



Another vector orthogonal to A and B, would, of course, be A x B and we could make it a unit vector by dividing by its magnitude. Thus, $u_2 = \dfrac{A \times B}{|A \times B|}$, but we wish u_2 to be in the plane determined by A and B, so that we really have found u_3! To obtain u_2, we just need a vector mutually orthogonal to u_1 and u_3 - namely, the cross-product, but we must be careful to insure that we preserve our right-handed coordinate system. If we observe the first figure above, we realize that to obtain u_2 in the correct orientation, we must find $u_3 \times u_1$. Therefore, we have, $u_1 = A/|A|$; $u_3 = \dfrac{A \times B}{|A \times B|}$, $u_2 = u_3 \times u_1$.

EXAMPLE 1: Construct a right-handed set of o-n vectors from $[4,-3,-12]$ and $[2,0,-1]$. Let $u_1 = [4,-3,-12]/13$; $u_3 = \dfrac{[4,-3,-12] \times [2,0,-1]}{|[4,-3,-12] \times [2,0,-1]|} = \dfrac{[3,-20,6]}{\sqrt{445}}$.

$$u_2 = u_3 \times u_1 = \dfrac{\begin{vmatrix} i & 3 & 4 \\ j & -20 & -3 \\ k & 6 & -12 \end{vmatrix}}{13\sqrt{445}} = \dfrac{[258,60,71]}{13\sqrt{445}}$$

58. GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

By the above we can see that given any two vectors we may construct an o-n set out of the given vectors. The next question which naturally arises is this. Can we construct an O-N set of n vectors in n-space to accomodate an(n + 1)th vector? i.e. can we, for example, construct an O-N set from two vectors in 2-space, from 3 vectors

in 3-space etc.? The answer is yes and the technique is called the <u>Gram-Schmidt orthogonalization process</u>. First, however, we might list general theorems regarding L.D and L.I. (without proofs) to justify and provide insight into the techniques used subsequently.

<u>Theorem 1</u>: If a set of vectors contains the null vector, then the set is L.D.

<u>Theorem 2</u>: If a set of vectors contains 2 identical vectors, the set is L.D.

<u>Theorem 3</u>: If a set of vectors is L.I., then any non-empty subset of the set is L.I.

Theorem 4: If a set of vectors is L.D., then any non-empty subset of the set is L.D.

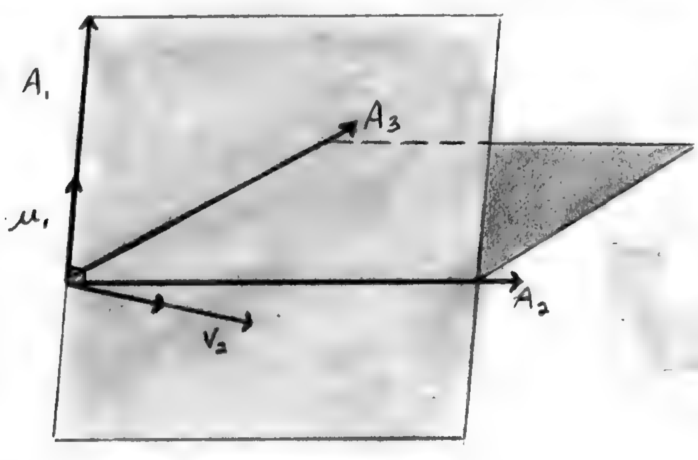
When we are able to express every vector in n-space in terms of n other L.I. vectors, we call the collection of L.I. vectors <u>generators</u> of the space. i.e. a set of vectors $A_1, A_2, \ldots\ldots A_n$ is called a set of generators if and only if every vector $B = x_1A_1 + x_2A_2 + \ldots\ldots\ldots x_nA_n$, where x_i is a scalar. Thus,

<u>Theorem 4</u> If $A_1, A_2, \ldots\ldots A_n$ is a set of generators and B is any vector, then the set $B, A_1, A_2, \ldots\ldots A_n$ is L.D.

<u>Theorem 5</u> Any 4 vectors are L.D. in 3-space and if 3 of them are L.I., they are a set of generators.

<u>Theorem 6</u>: If A_1, A_2, A_3 are L.I., then there always exists an O-N set u_1, u_2, u_3 such that u_1 is a scalar multiple of A_1, u_2 is a linear combination of A_1 and and A_2 and u_3 is a linear combination of A_1, A_2, A_3.

Theorem 6 justifies the technique now unfolded - namely the G-S process cited above. We proceed somewhat as before while referring to the illustrations.

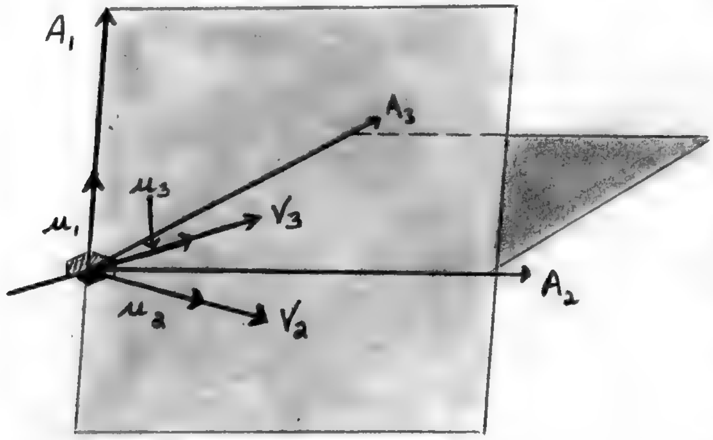


First, we unitize the vector in the A_1 direction. Thus, $u_1 = A_1/|A_1|$. Next, we construct a vector V_2 in the plane of u_1 and A_2 such that V_2 is perpendicular to u_1 and such that its magnitude will be [illegible]

$V_2 = A_2 + tu_1$. (We may do this because u_1, A_2, V_2 are L.D. and therefore V_2 may be expressed in terms of A_2 and u_1). But since V_2 is perpendicular to u_1, then $V_2 \cdot u_1 = 0$; i.e., $V_2 \cdot u_1 = (A_2 + tu_1) \cdot u_1 = 0$. Therefore, $A_2 \cdot u_1 + t(u_1 \cdot u_1) = 0$. This means that $t = -A_2 \cdot u_1$. Therefore, $V_2 = A_2 - (A_2 \cdot u_1)u_1$. To obtain u_2, we normalize V_2. Thus, $u_2 = V_2/|V_2|$.

We now have u_1 and u_2 and we may get u_3 by finding $u_1 \times u_2$ or we may proceed

in like manner as before. Consider the following illustration:



Now let V_3 be perpendicular to u_1 and V_2 and therefore to u_1 and u_2. Thus, u_1, u_2 and V_3 are L.I., but u_1, u_2, V_3, A_3 are L.D. (since they span 3-space). Thus, there exists an s, t such that $W_3 = A_3 + su_1 + tu_2$ by construction. But $V_3 \cdot u_1 = V_2 \cdot u_1 = 0$ (since they are mutually perpendicular). Therefore, $A_3 \cdot u_1 + s = 0$; $A_3 \cdot u_2 + t = 0$. Thus, we have, $V_3 = A_3 - (u_1 \cdot A_3)u_1 - (u_2 \cdot A_3)u_2$ and we unitize V_3 to obtain $u_3 = V_3/|V_3|$. The nice thing about the G-S process is that it may be extended to n-space. In 2-space we have: $u_1 = A_1/|A_1|$; $V_2 = A_2 - (u_1 \cdot A_2)u_1$. In 3-space we have: $u_1 = A_1/|A_1|$; $V_2 = A_2 - (u_1 \cdot A_2)u_1$; $V_3 = A_3 - (u_1 \cdot A_3)u_1 - (u_2 \cdot A_3)u_2$. In n-space, we have by induction: u_1 as above; V_2 as above; V_3 as above and $V_n = A_n - (u_1 \cdot A_n)u_1 - (u_2 \cdot A_n)u_2 - \dots\dots\dots\dots - (u_{n-1} \cdot A_n)u_{n-1}$ where $u_i = V_i/|V_i|$, $i = 2, 3..$ Note, however, that this derivation does not guarantee a right-handed system - only an O-N base!

<u>EXAMPLE 1</u>: Construct a right-handed O-N set from the vectors $[4,-3,-12]$, $[2,0,-1]$, $[-3,2,1]$

$u_1 = [4,-3,-12]/13$. $V_2 = [2,0,-1] - \left(\frac{[4,-3,-12]}{13} \cdot [2,0,-1]\right)\frac{[4,-3,-12]}{13}$.

Thus, $V_2 = [2,0,-1] - 20/13^2[4,-3,-12] = \frac{[258,60,71]}{169}$ and $u_2 = \frac{[258,60,71]}{13\sqrt{445}}$ (we may disregard 169 since we are unitizing anyway!

$$V_3 = [-3,2,1] - \left(\frac{[4,-3,-12]}{13} \cdot [-3,2,1]\right)\frac{[4,-3,-12]}{13} - \left(\frac{[258,60,71]}{13\sqrt{445}} \cdot [-3,2,1]\right)\frac{[258,60,71]}{13\sqrt{445}}$$

$$= [-3,2,1] + \frac{30[4,-3,-12]}{169} + \frac{583}{169(445)}[258,60,71] = \frac{[-21801,145340,-43602]}{169(445)}$$

$= \frac{[-3,20,-6]7267}{169(445)}$ and $u_3 = \frac{[3,-20,6]}{\sqrt{445}}$ for right-handed system! Note that u_3 could be calculated much more easily from u_1 and u_2. That is, $u_1 \times u_2 = \frac{[4,-3,-12]}{13} \times \frac{[258,60,71]}{13\sqrt{445}}$

$= \frac{[507,-3380,1014]}{169\sqrt{445}} = \frac{[3,-20,6]}{\sqrt{445}}$.

59. <u>EXPRESSION OF VECTORS IN TERMS OF A GIVEN O-N BASE</u>

The proof of the following formula is omitted but quite straight-forward. We may

express any arbitrary vector $A = [A_1, A_2, \ldots\ldots, A_n]$ in n-space for example, in terms of unit vectors u_1, u_2, ..., u_n (a given orthonormal set) by the expression:

$A = (A \cdot u_1)u_1 + (A \cdot u_2)u_2 + \ldots\ldots\ldots + (A \cdot u_n)u_n$.

EXAMPLE 1: Given the O-N set of examples 1, articles 57 and 58, express $[-5,3,2]$ in terms of this set.

$A = [-5,3,2]$ and $u_1 = \dfrac{[4,-3,-12]}{13}$; $u_2 = \dfrac{[258,60,71]}{13\sqrt{445}}$; $u_3 = \dfrac{[3,-20,6]}{\sqrt{445}}$.

We have, $A \cdot u_1 = -53/13$; $A \cdot u_2 = -968/13\sqrt{445}$; $A \cdot u_3 = -63/\sqrt{445}$. Therefore:

$$[-5,3,2] = -53/13\dfrac{[4,-3,-12]}{13} - \dfrac{968}{13\sqrt{445}}\,\dfrac{[258,60,71]}{13\sqrt{445}} - \dfrac{63}{\sqrt{445}}\,\dfrac{[3,-20,6]}{\sqrt{445}}.$$

This result can easily be verified by the student. Note that if the system is not O-N, the procedure is much more difficult since algebraic cross-product terms will be involved.

60. RECIPROCAL SETS OF VECTORS

Sometimes one is interested in finding a reciprocal set of vectors(i.e. a perpendicular set to the given L.I. set used as a basis). This, of course, is unnecessary when dealing with O-N bases, since any orthogonal set will be its own reciprocal. With oblique systems, however, we may have two sets of mutually perpendicular L.I. vectors and the formula for these vectors in 3-space is:

$A' = \dfrac{B \times C}{A \cdot B \times C}$; $B' = \dfrac{C \times A}{A \cdot B \times C}$; $C' = \dfrac{A \times B}{A \cdot B \times C}$ where A,B,C are L.I. and form a basis for 3-space.

EXAMPLE 1: Find a set of vectors reciprocal to the set $[2,3,-1]$; $[1,-1,-2]$; $[-1,2,2]$.

$$A \cdot B \times C = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ -1 & -2 & 2 \end{vmatrix} = 3; \quad \begin{aligned} A' &= \dfrac{[1,-1,-2] \times [-1,2,2]}{3} = [2,0,1]/3 \\ B' &= \dfrac{[-1,2,2] \times [2,3,-1]}{3} = [-8,3,-7]/3 \\ C' &= \dfrac{[2,3,-1] \times [1,-1,-2]}{3} = [-7,3,-5]/3 \end{aligned}$$

Here A is perpendicular to B', B perpendicular to C' and C perpendicular to A' as can be checked by the dot product.

EXERCISES(25): Given the vectors $A[-4,3,12]$; $B[3,-2,6]$; $C[-6,1,18]$.

1. Express $[1,2,3]$ in terms of A,B,C.
2. Construct an O-N set out of A,B.
3. " " " " " " A,C.
4. " " " " " " B,C.
5. " " " " " " A,B,C.

6. Construct a reciprocal set out of the vectors A,B,C.

7. Construct an O-N set out of [1,-2,2]; [2,-2,1]; [2,1,-2].

8. Construct another O-N set out of the above vectors in 7.

9. Construct still another set of O-N vectors out of the set given in exercise 7.

10. Express A above in terms of the O-N set found in exercise 7,8 or 9.

CHAPTER 11 - ROTATION & TRANSLATION IN 2 & 3-SPACE BY USE OF VECTORS AND MATRICES

61. EIGENVECTORS AND EIGENVALUES

In order to see how matrices will help us in our quest to simplify the general equation of the 2nd degree in 2 or 3-space(i.e. removing cross-product terms), we must recall the formulas for rotation in 2-space. They were:

$x = x'\cos\theta - y'\sin\theta$; $y = x'\sin\theta + y'\cos\theta$. Since these are linear equations, we may put these in matrix form to give: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

Any quadratic form may be put into matrix notation, for example, $Ax^2 + 2Bxy + Cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Now there are some interesting things to notice about this form. First, $\begin{bmatrix} x, & y \end{bmatrix}$ is the transpose of $\begin{bmatrix} x \\ y \end{bmatrix}$ and second, $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is a symmetric matrix. Again, in 3-space, the quadratic form:

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fy^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and we see that $\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}'$ and $\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}$ is symmetric as in the case of 2-space.

Now let us call the unprimed system of coordinates the X system and the primed system of coordinates the Y system for the moment. Then if we were to call the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = P$, we would have a vector equation which would say that $X = PY$. Further, if we were to name the coefficient matrix $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ or $\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}$

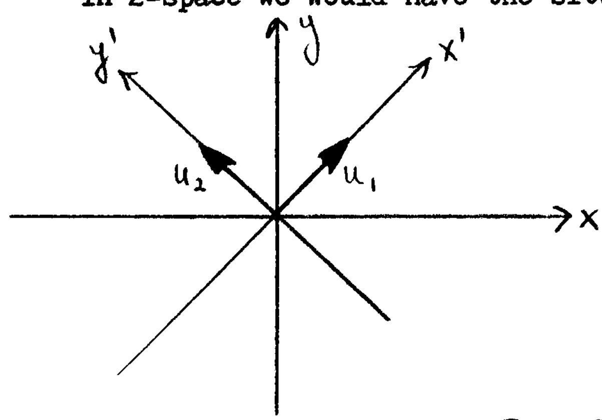
the "A" matrix, our quadratic form in matrix notation would become $X'AX$(where X still represents the unprimed system). But from the relation $X = PY$, we would have:

$X'AX = (PY)'A(PY) = (Y'P')A(PY) = Y'(P'AP)Y.$

Now if we were to call the matrix found by $P'AP$ our "B" matrix, we can quickly see that we have $Y'BY$, which is another quadratic form - but this time, in the primed system. The only thing we want to make sure of is that the B matrix has no cross-product terms, i.e. it is a diagonal matrix. In 2-space we might have $B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ where λ_1 and λ_2 are scalars and the coefficients of the x' and y' terms after rotation, i.e.

$\lambda_1 x'^2 + \lambda_2 y'^2 = [x' \; y'] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

In 2-space we would have the situation depicted below in the illustration:



We require vectors u_1, u_2(unit vectors) that are invariant under rotation, i.e. we want to find vectors whose components are $[x,y]$ which are directionally unchanged under the transformation since we can always unitize a vector of any magnitude.

Mathematically, we would have: $\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ This is equivalent to the system of equations:

$$\begin{matrix} Ax + By = \lambda x \\ Bx + Cy = \lambda y \end{matrix} \quad \text{This gives:} \quad \begin{matrix} Ax - \lambda x + By = 0 \\ Bx + Cy - \lambda y = 0 \end{matrix} \quad \text{Hence,} \quad \begin{matrix} x(A - \lambda) + By = 0 \\ Bx + y(C - \lambda) = 0 \end{matrix}$$

Now to obtain a non-trivial solution, we recall that the determinant of the coefficient matrix must equal zero. This yields $\lambda^2 + (A - C)\lambda + (B^2 - AC) = 0$ and we may find 2 values of λ that satisfy the equation. These values are called <u>eigenvalues</u> or <u>characteristic values</u> and the values $[x,y]$ obtained by back substituting into the original equations are called <u>eigenvectors</u> or <u>characteristic vectors</u>. These vectors are vectors which remain invariant under the particular transformation of coordinates and this provides us with information about the equations of rotation and the final form of the result after rotation.

<u>EXAMPLE 1</u>: Find eigenvectors and eigenvalues for the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

From the above, we have, $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$. Thus, $-4 - 3\lambda + \lambda^2 = 0$. $\lambda = 4,-1$

To obtain the eigenvectors we have the two equations:

$$\left.\begin{aligned} x(1-4) + 2y &= 0 \\ 3x + (2-4)y &= 0 \end{aligned}\right\}$$ 1 solution is (2,3); therefore the eigenvector is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

We write vectors in column form to agree with the above discussion. Also, we have,

$$\left.\begin{aligned} x(1+1) + 2y &= 0 \\ 3x + (2+1)y &= 0 \end{aligned}\right\}$$ A solution is (-1,1); therefore the eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Of course any scalar multiple of these vectors are also eigenvectors, but we choose the simplest.

62. <u>RELATIONSHIP OF EIGENVECTORS AND EIGENVALUES TO ROTATION IN 2 AND 3-SPACE</u>

As stated above we require a diagonal matrix B whose coefficients are exactly those required for the primed system(i.e.after rotation). There are several pertinent theorems that allow us to find these coefficients. These are listed below without proof:

<u>Theorem 1</u>: Every real symmetric matrix A is similar to a diagonal matrix B whose diagonal elements are the eigenvalues of A.

<u>Theorem 2</u>: A is similar to B if and only if there is a matrix P where $P \neq 0$ & $B = P^{-1}AP$.

<u>Theorem 3</u>: Similar matrices have the same eigenvalues.

<u>Theorem 4</u>: Every real symmetric matrix is similar to a diagonal matrix.

<u>Theorem 5</u>: If A is a real n-square symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$, then there always exists a real orthogonal matrix P such that; $P'AP = P^{-1}AP = \text{diag}\,(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$.

<u>Definition</u>: A is orthogonal if and only if $AA' = I$, i.e., $A' = A^{-1}$.

These theorems and definitions tell us then that the eigenvalues we find will be our final coefficients that we needed in our B matrix and that we may obtain our equations of rotation from the P matrix which must always exist whenever there are eigenvalues. Also, the P matrix is composed of the eigenvectors found. Some examples might help to clarify these concepts.

<u>EXAMPLE 1</u>: Simplify $5x^2 + 8xy + 5y^2 - 9 = 0$ by a suitable rotation.

First we obtain our A matrix: $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. Then we find eigenvalues: $\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$.

Thus $25 - 10\lambda + \lambda^2 - 16 = 0$, i.e. $\lambda^2 - 10\lambda + 9 = 0$. Thus $\lambda = 1,9$.

Since $\lambda = 1,9$ then our final form must be either $x^2 + 9y^2 = 9$ or $9x^2 + y^2 = 9$. To find eigenvectors and our P matrix, we solve:

$$\left.\begin{aligned} 4x + 4y &= 0 \\ 4x + 4y &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \left.\begin{aligned} -4x + 4y &= 0 \\ 4x - 4y &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We form our P matrix keeping in mind the form of the rotation equations with regard to the sign, i.e., $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$. Thus we have $P = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}}$ and $P^{-1} = P' = \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}}$.

Note that we must unitize the vectors and therefore we divide by $\sqrt{2}$.

To check, we form $P^{-1}AP$ and perform the indicated operations:

$$\frac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \frac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix} 9 & 9 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} \frac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}}{2} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore our final form is $9x'^2 + y'^2 = 9$ or $\dfrac{x'^2}{1} + \dfrac{y'^2}{9} = 1.$ The equations of rotation are taken from the P matrix and we have: $x = \dfrac{x' - y'}{\sqrt{2}}$; $y = \dfrac{x' + y'}{\sqrt{2}}$. Compare this result with that of example 1, article 8.

EXAMPLE 2: Simplify $4x^2 + 12xy - y^2 + 80$ by a suitable rotation.

$A = \begin{bmatrix} 4 & 6 \\ 6 & -1 \end{bmatrix}$ and $\begin{bmatrix} 4 - \lambda & 6 \\ 6 & -1 - \lambda \end{bmatrix} = 0.$ Thus, $-4 - 3\lambda + \lambda^2 - 36 = 0$, thus $\lambda^2 - 3\lambda - 40 = 0$ and $\lambda = 8, -5$.

Our final form must be $8x'^2 - 5y'^2 = -80$ or $5x'^2 - 8y'^2 = +80$. The eigenvectors are:

$$\left.\begin{aligned} -4x + 6y &= 0 \\ 6x - 9y &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \left.\begin{aligned} 9x + 6y &= 0 \\ 6x + 4y &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Therefore, $P = \dfrac{\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}}{\sqrt{13}}$ $\quad P^{-1} = \dfrac{\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}}{\sqrt{13}}$ $\quad P^{-1}AP = \dfrac{\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}}{\sqrt{13}\cdot\sqrt{13}}$

$$= \frac{\begin{bmatrix} 24 & 16 \\ 10 & -15 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}}{\sqrt{13}\cdot\sqrt{13}} = \frac{\begin{bmatrix} 104 & 0 \\ 0 & -65 \end{bmatrix}}{13} = \begin{bmatrix} 8 & 0 \\ 0 & -5 \end{bmatrix}.$$

Thus, $8x'^2 - 5y'^2 = -80$ or $\dfrac{x'^2}{10} - \dfrac{y'^2}{16} = -1.$

The equations of rotation are: $x = \dfrac{3x' - 2y'}{\sqrt{13}}$; $y = \dfrac{2x' + 3y'}{\sqrt{13}}$. Compare this result with example 2, article 8.

EXAMPLE 3: Simplify $5x^2 + 3y^2 + 3z^2 + 2xy - 2xz - 2yz - 6 = 0$ by a suitable rotation.

$A = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 5 - \lambda & 1 & -1 \\ 1 & 3 - \lambda & -1 \\ -1 & -1 & 3 - \lambda \end{bmatrix} = 0.$ Solving for λ, we obtain,

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

Using the theory of equations and synthetic division to obtain the roots, we have,

$$\begin{array}{l} 1 - 11 + 36 - 36 \; \lfloor 3 \\ \underline{\;\; + \; 3 - 24 + 36} \\ 1 - \; 8 + 12 \; \lfloor 2 \\ \underline{\;\; + \; 2 - 12} \\ 1 - \; 6 \; \lfloor 6 \end{array}$$

Hence, $\lambda = 2,3,6$ and the final form will be; $2x^2 + 3y^2 + 6z^2 = 6$ or some variable permutation of this, e.g., $6x^2 + 3y^2 + 2z^2 = 6$. In any case, we see immediately that the conicoid in question is an ellipsoid.

Eigenvectors:

$$\left.\begin{array}{r} 3x + y - z = 0 \\ x + y - z = 0 \\ -x - y + z = 0 \end{array}\right\} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \left.\begin{array}{r} 2x + y - z = 0 \\ x \quad - z = 0 \\ -x - y \quad = 0 \end{array}\right\} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \left.\begin{array}{r} -x + y - z = 0 \\ x - 3y - z = 0 \\ -x - y - 3z = 0 \end{array}\right\} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

The P matrix is: $\underset{\sqrt{2}\sqrt{3}\sqrt{6}}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}$ and P^{-1} is: $\underset{\sqrt{2}\sqrt{3}\sqrt{6}}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}}$

Note that we have been "carrying along" the $\sqrt{}$ and this is justified since if we actually perform the surd mutltiplication with the P matrix and P^{-1} Matrix, we have:

$$\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 0 & 2/\sqrt{2} & 2/\sqrt{2} \\ 3/\sqrt{3} & -3/\sqrt{3} & 3/\sqrt{3} \\ 12/\sqrt{6} & 6/\sqrt{6} & -6/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

If we work with integers only, we have:

$$\underset{\sqrt{2}\sqrt{3}\sqrt{6}}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}} \begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \underset{\sqrt{2}\sqrt{3}\sqrt{6}}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}} = \underset{2:3:6}{\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 3 \\ 12 & 6 & -6 \end{bmatrix}} \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \underset{2:3:6}{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 36 \end{bmatrix}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

where the surds are "carried along" as ratios - this greatly eases the computation. The equations of rotation are:

$x = y'/\sqrt{3} + 2z'/\sqrt{6}$; $y = x'/\sqrt{2} - y'/\sqrt{3} + z'/\sqrt{6}$; $z = x'/\sqrt{2} + y'/\sqrt{3} - z'/\sqrt{6}$. Substitution in the original equation will yield the final result $2x'^2 + 3y'^2 + 6z'^2 = 6$.

63. ROTATION AND TRANSLATION IN 2 AND 3-SPACE BY USE OF MATRICES

The combined operations of rotation and translation may be accomplished by considering the complete 2nd degree equation in 2-space, for example, in matrix form.

$Ax^2 + 2Bxy + Cy^2 + Dx + Ey = F$; i.e. $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = F.$

If we let $\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x' \\ y' \end{bmatrix}$ then $[x \quad y] = [x' \quad y']P'$(taking the transpose of both sides).

Substituting, we have, $[x' \quad y']P'\begin{bmatrix} A & B \\ B & C \end{bmatrix} P \begin{bmatrix} x' \\ y' \end{bmatrix} + [D \quad E]P\begin{bmatrix} x' \\ y' \end{bmatrix} = F$. But this means:

$$[x' \quad y']\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + [D \quad E]P\begin{bmatrix} x' \\ y' \end{bmatrix} = F.$$

We proceed as before, but it might be mentioned that a combination of partial derivatives and matrices is probably more propitious even with the parabola or paraboloid since we can always back substitute using the P^{-1} matrix equations instead of the P matrix equations. An example should show the difference.

<u>EXAMPLE 1</u>: Simplify $x^2 + 2xy + y^2 + 4x - 4y = 4$ by a suitable rotation.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$. Thus, $\lambda = 0,2$

<u>Eigenvectors</u>: $\left.\begin{matrix} x + y = 0 \\ x + y = 0 \end{matrix}\right\} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \left.\begin{matrix} -x + y = 0 \\ x - y = 0 \end{matrix}\right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad P = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \quad P^{-1} = \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}}$

Thus, we have, $[x' \quad y']\dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} x' \\ y' \end{bmatrix} + \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} x' \\ y' \end{bmatrix} = 4$. Therefore,

$[x' \quad y']\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + \dfrac{[0 \quad -8]}{\sqrt{2}}\begin{bmatrix} x' \\ y' \end{bmatrix} = 4$; hence, $2x'^2 - 4\sqrt{2}y' = 4$.

However, using the P^{-1} matrix, we have $x' = \dfrac{x + y}{\sqrt{2}}$; $y' = \dfrac{-x + y}{\sqrt{2}}$ and since $x^2 + 2xy + y^2 = (x + y)^2 = 2x'^2$ and $4x - 4y = -4(-x + y)$, we therefore have: $2x'^2 - 4\sqrt{2}y' = 4$.

It is indeed fortuitious that $4x - 4y$ yielded the equation in terms of y', but even if it didn't, we could use partial differentiation or completion of the square once the substitution for x and y had been made to find the point of translation.

In 3-space the matrix method would yield:

$$[x \quad y \quad z]\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + [G \quad H \quad I]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 + Gx + Hy + Iz = J.$$

Letting $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ we have $[x \quad y \quad z] = [x' \quad y' \quad z']P'$ and substituting we get:

$$[x' \quad y' \quad z']P'\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + [G \quad H \quad I]P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

<u>EXAMPLE 2</u>: Simplify $(2x - 2y - 3z)^2 + 19x - 34y - 44z + 50 = 0$ by a suitable rotation and translation.

$$A = \begin{bmatrix} 4 & -4 & -6 \\ -4 & 4 & 6 \\ -6 & 6 & 9 \end{bmatrix} \text{ and } \begin{vmatrix} 4-\lambda & -4 & -6 \\ -4 & 4-\lambda & 6 \\ -6 & 6 & 9-\lambda \end{vmatrix} = 0. \text{ Thus } \lambda^2(17 - \lambda) = 0 \text{ and } \lambda = 0,0,17.$$

<u>Eigenvalues</u>:
$$\left.\begin{aligned} 4x - 4y - 6z &= 0 \\ -4x + 4y + 6z &= 0 \\ -6x + 6y + 9z &= 0 \end{aligned}\right\} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad (\lambda = 0) \qquad \left.\begin{aligned} -13x - 4y - 6z &= 0 \\ -4x - 13y + 6z &= 0 \\ -6x + 6y - 8z &= 0 \end{aligned}\right\} \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} \quad (\text{when } \lambda = 17)$$

Now we may get 3rd vector by taking the cross-product of the above vectors! Thus,

$$\begin{bmatrix} i & 1 & 2 \\ j & -2 & -2 \\ k & 2 & -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 2 \end{bmatrix} \qquad P = \begin{bmatrix} 1/3 & 2/\sqrt{17} & 10/\sqrt{153} \\ -2/3 & -2/\sqrt{17} & 7/\sqrt{153} \\ 2/3 & -3/\sqrt{17} & 2/\sqrt{153} \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/\sqrt{17} & -2/\sqrt{17} & -3/\sqrt{17} \\ 10/\sqrt{153} & 7/\sqrt{153} & 2/\sqrt{153} \end{bmatrix}$$

Now we have two sets of rotation equations:

$$x = x'/3 + 2y'/\sqrt{17} + 10z'/\sqrt{153}; \; y = -2x'/3 - 2y'/\sqrt{17} + 7z'/\sqrt{153}; \; z = 2x'/3 - 3y'/\sqrt{17} + \frac{2z'}{\sqrt{153}}$$

$$x' = \frac{x - 2y + 2z}{3}; \; y' = \frac{2x - 2y - 3z}{\sqrt{17}}; \; z' = \frac{10x + 7y + 2z}{\sqrt{153}}$$

and appropriate substitution of these equations into the original equation yields:

$17y'^2 + 19(x'/3 + 2y'/\sqrt{17} + 10z'/\sqrt{153}) - 34(-2x'/3 - 2y'/\sqrt{17} + 7z'/\sqrt{153}) - 44(2x'/3 - 3y'/\sqrt{17} + 2z'/\sqrt{153}) + 50 = 0$, which reduces to: $17y'^2 - x'/3 + 238y'/\sqrt{17} - 136z'/\sqrt{153} + 50 = 0$. We may get rid of the 'x' or 'z' term by using the equations of rotation $x' = x''\cos\theta - z''\sin\theta$ and $z' = x''\sin\theta + z''\cos\theta$. Substituting and rearranging terms, we get: $17y'^2 + 238y'/\sqrt{17} - x''(\cos\theta/3 + 136\sin\theta/\sqrt{153}) + z''(\sin\theta/3 - 136\cos\theta/\sqrt{153}) + 50 = 0$. Now let either $(\cos\theta/3 + 136\sin\theta/\sqrt{153})$ or $(\sin\theta/3 - 136\cos\theta/\sqrt{153}) = 0$. We choose the latter, thus z'' drops out and $\tan\theta = \sin\theta/\cos\theta = 136\cdot 3/\sqrt{153} = 8\sqrt{17}$. Thus $\sin\theta = 8\sqrt{17}/33$ and $\cos\theta = 1/33$. This yields the final form: $17(y' + 7/\sqrt{17})^2 = 11(x'' - 1/11)$. By the matrix method:

$$[x' \quad y' \quad z']\begin{bmatrix} 0 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + [19 \quad -34 \quad -44]\begin{bmatrix} 1/3 & 2/\sqrt{17} & 10/\sqrt{153} \\ -2/3 & -2/\sqrt{17} & 7/\sqrt{153} \\ 2/3 & -3/\sqrt{17} & 2/\sqrt{153} \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + 50 = 0.$$

Thus, $17y'^2 + [-1/3 \quad 238/\sqrt{17} \quad -136/\sqrt{153}]\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + 50 = 17y'^2 - x'/3 + \frac{238y'}{\sqrt{17}} - \frac{136z'}{\sqrt{153}} + 50 = 0,$

and we proceed as before!

EXAMPLE 3: Simplify $5x^2 + 7y^2 + 6z^2 - 4yz - 4xz - 6x - 10y - 4z + 7 = 0$ by a suitable rotation and translation.

$$\frac{\partial F}{\partial x} = 10x - 4z - 6 = 0 \qquad \begin{bmatrix} 5 & 0 & -2 & 3 \\ 0 & 7 & -2 & 5 \\ -1 & -1 & 3 & 1 \end{bmatrix} \overset{R_1+R_3}{\sim} \begin{bmatrix} 5 & 0 & -2 & 3 \\ 0 & 7 & -2 & 5 \\ 0 & -1 & 13/5 & 8/5 \end{bmatrix} \overset{7R_3+R_2}{\sim} \begin{bmatrix} 5 & 0 & -2 & 3 \\ 0 & 0 & 81/5 & 81/5 \\ 0 & -1 & 13/5 & 8/5 \end{bmatrix}$$

$$\frac{\partial F}{\partial y} = 14y - 4z - 10 = 0$$

$$\frac{\partial F}{\partial z} = -4x - 4y + 12z - 4 = 0 \qquad \underset{-\frac{13}{81}R_2+R_3}{\overset{R_2 \div 81/5}{\sim}} \begin{bmatrix} 5 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

Thus, $x = 1$, $y = 1$, $z = 1$.

Using the translation equations and dropping primes, we have:

$$5(x+1)^2 + 7(y+1)^2 + 6(z+1)^2 - 4(y+1)(z+1) - 4(x+1)(z+1) - 6(x+1) - 10(y+1) - 4(z+1) + 7 = 0.$$

This reduces to: $5x^2 + 7y^2 + 6z^2 - 4yz - 4xz - 3 = 0.$

$$A = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 7 & -2 \\ -2 & -2 & 6 \end{bmatrix} \text{ and } \begin{vmatrix} 5-\lambda & 0 & -2 \\ 0 & 7-\lambda & 0 \\ -2 & -2 & 6-\lambda \end{vmatrix} = 0.$$

Thus $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$ Therefore, by synthetic division and the theory of equations, we have

$$\begin{array}{l|l} 1 - 18 + 99 - 162 & 3 \\ \quad\;\; 3 - 45 + 162 & \\ \hline 1 - 15 + 54 & 6 \\ \quad\;\; 6 - 54 & \\ \hline 1 - 9 & \end{array}$$

Thus, $\lambda = 3, 6, 9$ and the final form is $3x^2 + 6y^2 + 9z^2 - 3 = 0.$

Eigenvectors:

$$\left.\begin{aligned} 2x \qquad - 2z &= 0 \\ 4y - 2z &= 0 \\ -2x - 2y + 3z &= 0 \end{aligned}\right\} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \left.\begin{aligned} -x \qquad - 2z &= 0 \\ y - 2z &= 0 \\ -2x - 2y \qquad &= 0 \end{aligned}\right\} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \quad \left.\begin{aligned} -4x \qquad - 2z &= 0 \\ -2y - 2z &= 0 \\ -2x - 2y - 3z &= 0 \end{aligned}\right\} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$P = \dfrac{\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix}}{3} \qquad P^{-1} = \dfrac{\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{bmatrix}}{3}$$

Now

$$\dfrac{\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{bmatrix}}{3} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 7 & -2 \\ -2 & -2 & 6 \end{bmatrix} \dfrac{\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix}}{3} =$$

$$\dfrac{\begin{bmatrix} 6 & 3 & 6 \\ 12 & -12 & -6 \\ 9 & 18 & -18 \end{bmatrix}}{3} \dfrac{\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix}}{3} = \dfrac{\begin{bmatrix} 27 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 81 \end{bmatrix}}{9} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Another method would be:

$$[x' \;\; y' \;\; z']P' \begin{bmatrix} 5 & 0 & -2 \\ 0 & 7 & -2 \\ -2 & -2 & 6 \end{bmatrix} P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + [-6, -10, -4] \dfrac{\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix}}{3} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + 7 = 0.$$

We have, $3x'^2 + 6y'^2 + 9z'^2 - 10x' + 4y' - 6z' + 7 = 0.$

The new origin is found to be at (5/3,-1/3,1/3) which is different from (1,1,1) as expected since in this case, rotation takes place first - then translation. Hence, the coordinates with respect to the original axes must be different.

EXERCISES(26):

1. Obtain the results in exercise(3) by using eigenvectors and eigenvalues.
2. Do the same for the problems given in exercise(4) and (5).

Simplify the following by a suitable rotation and translation:

3. $7x^2 + y^2 + z^2 + 16yz + 8xz - 8xy + 2x + 4y - 40z - 14 = 0.$
4. $2x^2 - y^2 - 10z^2 + 20yz - 8xz - 28xy + 16x + 26y + 16z - 34 = 0.$
5. $11x^2 + 5y^2 + 2z^2 + 20yz - 16xy + 4xz - 10x - 14y - 28z + 26 = 0.$
6. $4x^2 + 3y^2 + 2z^2 + 4yz - 4xy - 4x - 6y - 8z + 6 = 0.$
7. $x^2 - y^2 + 4yz - 6x - 2y - 8z + 5 + 4xz = 0$
8. $32x^2 + y^2 + z^2 + 6yz - 16xz - 16xy - 6x - 12y - 12z + 18 = 0.$
9. $5x^2 - 16y^2 + 5z^2 + 8yz - 14xz + 8xy + 4x + 20y + 4z - 24 = 0.$
10. $7x^2 + 33y^2 + 7z^2 + 12yz - 10xz - 12xy - 36 = 0.$
11. $x^2 - 31y^2 + z^2 - 20yz - 6xz + 20xy - 36 = 0.$
12. $2x^2 + 33y^2 + 2z^2 + 12yz - 20xz - 12xy - 72 = 0.$
13. $13x^2 + 20y^2 + 5z^2 - 12yz + 14xz - 4xy - 336 = 0.$
14. $5x^2 - 12y^2 - 3z^2 + 20yz - 2xz + 28xy - 336 = 0.$
15. $2x^2 + 5y^2 + 10z^2 + 12yz + 6xz + 4xy - 1 = 0.$

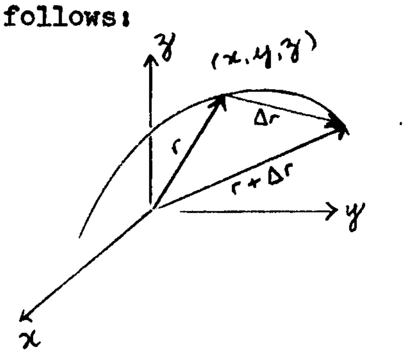
CHAPTER 12 -VECTOR DIFFERENTIATION

64. DIFFERENTIATION OF A VECTOR WITH RESPECT TO A SCALAR

This type of differentiation concerns itself with vector functions that are expressed in terms of some scalar parameter "t" which is allowed to take on different values. For example, we might have parametric equations of the circle $x^2 + y^2 = 4$ in 2-space and we might wish to find the tangent vector to the curve. First, it might be instructive to express some conics in parametric form. The aforementioned circle $x2 + y^2 = 4$ in parametric form would be expressed as $x = 2\cos t$; $y = 2\sin t$. It can quickly be seen that by squaring both sides of the individual equations and then adding them, the result will be $x^2 + y^2 = 4$. The conic $b^2x^2 + a^2y^2 = a^2b^2$ can be expressed in parametric form as $x = a\cos t$; $y = b\sin t$. Here, squaring and subsequent

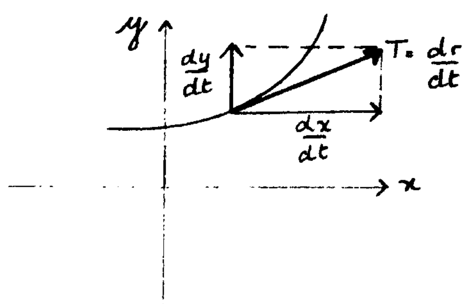
multiplication of the 1st equation by b^2 and the second by a^2 and then adding will give the desired result. The hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be expressed as $x = a\sec t$; $y = b\tan t$.

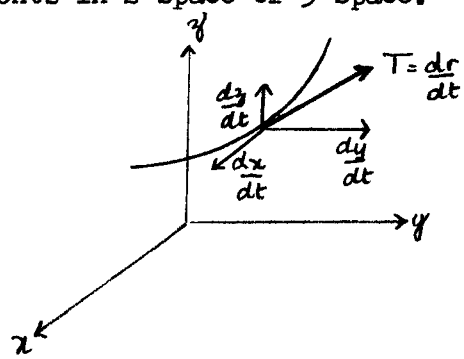
In 3-space, equations in parametric form are extremely useful since they can describe a particular curve in space not necessarily a surface. In either 2-space or 3-space then, we might have some curve described by parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$. The position vector $\vec{r}$ to any point in 2-space will be $\vec{r} = [x,y] = [f(t),g(t)]$ and in 3-space wil be $\vec{r} = [x,y,z] = [f(t),g(t),h(t)]$. We are particularly interested in finding the tangent vector to the curve and we define this in the usual way, i.e. $\lim\limits_{\Delta t \to 0} \dfrac{r(t + \Delta t) - r(t)}{\Delta t} = \dfrac{dr}{dt}$ if the limit exists. Geometrically, we can depict the situation as follows:



$r + \Delta r$ is given by $r(t + \Delta t)$, of course, and $r(t) = r$. Thus, we find that $dr/dt = [f'(t),g'(t),h'(t)]$. Sometimes the notation $\dot{r}$, $\dot{x}$, $\dot{y}$, etc., is used to denote dr/dt, dx/dt, dy/dt(derivative with respect to t).

Note that we may resolve $\dot{r}$ into its components in 2-space or 3-space.





Since $dr/dt = [f'(t),g'(t)]$ in 2-space, we have $dr/dt = [dx/dt,dy/dt]$ and in 3-space we have $dr/dt = [dx/dt,dy/dt,dz/dt]$. The short forms are $\dot{r} = [\dot{x},\dot{y}]$ for 2-space and $\dot{r} = [\dot{x},\dot{y},\dot{z}]$ for 3-space.

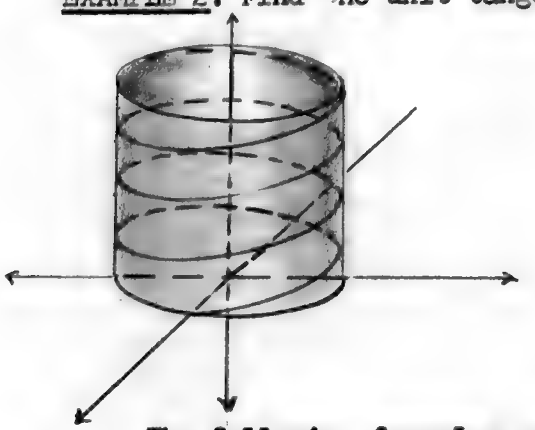
An interesting situation develops when the parameter is considered to be the arc length s, because then we have $dr/ds = [dx/ds,dy/ds]$ in 2-space and $dr/ds = [dx/ds,dy/ds,\frac{dz}{ds}]$ in 3-space. Now if we find the magnitude of dr/ds, we see that $|dr/ds| = \sqrt{(dx/ds)^2 + (dy/ds)^2} = \sqrt{ds^2/ds^2} = 1$ in 2-space and $|dr/ds| = \sqrt{(dx/ds)^2 + (dy/ds)^2 + (dz/ds)^2} = 1$ in 3-space. In other words, dr/ds is a <u>unit tangent vector</u> to a particular curve at the point (x,y) or (x,y,z)

Since dr/dt is simply a tangent vector to a curve at any point, to obtain a unit tangent vector, we merely normalize it by dividing by the magnitude. Therefore, we have, $dr/ds = \frac{dr/dt}{|dr/dt|}$. We denote dr/ds, the unit tangent vector by τ (tau).

Recall that by the chain rule $dr/dt = dr/ds \cdot ds/dt$. Substituting into the expression for dr/ds above, we have $dr/ds = \frac{dr/ds \cdot ds/dt}{|dr/dt|}$ and thus, $|dr/dt| = ds/dt$, a relationship that will prove useful.

<u>EXAMPLE 1</u>: Find the unit tangent vector to the curve x = cost, y = sint at any point (x,y). $r = [x,y] = [\cos t, \sin t]$ and hence, $\dot{r} = [-\sin t, \cos t]$ and $\dot{r}/|\dot{r}| = \frac{[-\sin t, \cos t]}{\sqrt{\sin^2 t + \cos^2 t}} = [-\sin t, \cos t]$.

<u>EXAMPLE 2</u>: Find the unit tangent vector to the circular helix x = acost, y = asint, z = at.



Now $r = [a\cos t, a\sin t, at]$ and therefore, $\dot{r} = [-a\sin t, a\cos t, a]$. $dr/ds = \dot{r}/|\dot{r}| =$

$\frac{[-a\sin t, a\cos t, a]}{\sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2}}$ Simplifying, we get,

$\frac{[-\sin t, \cos t, 1]}{\sqrt{2}}$

The following formulas may be readily established by application of the definitions given above for differentiation. The symbol "ϕ" (phi) denotes a differentiable <u>scalar</u> function of t.

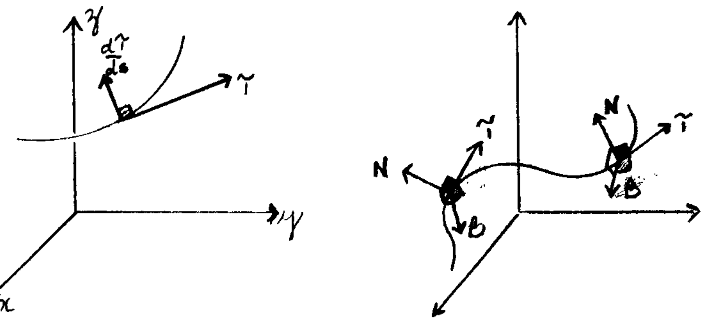
1. $d/dt(\vec{A} + \vec{B}) = d\vec{A}/dt + d\vec{B}/dt$.
2. $d/dt(\vec{A} \cdot \vec{B}) = \vec{A} \cdot d\vec{B}/dt + \vec{B} \cdot d\vec{A}/dt$.
3. $d/dt(\vec{A} \times \vec{B}) = \vec{A} \times d\vec{B}/dt + \vec{B} \times d\vec{A}/dt$.
4. $d/dt(\phi\vec{A}) = \phi d\vec{A}/dt + \vec{A} d\phi/dt$.
5. $d/dt(\vec{A} \cdot \vec{B} \times \vec{C}) = \vec{A} \cdot \vec{B} \times d\vec{C}/dt + \vec{A} \cdot d\vec{B}/dt \times \vec{C} + d\vec{A}/dt \cdot \vec{B} \times \vec{C}$.
6. $d/dt[\vec{A} \times (\vec{B} \times \vec{C})] = \vec{A} \times (\vec{B} \times d\vec{C}/dt) + \vec{A} \times (d\vec{B}/dt \times \vec{C}) + dA/dt \times (B \times C)$.

<u>EXAMPLE 3</u>: Find the unit tangent vector to the twisted cubic $x = at$, $y = bt^2$, $z = ct^3$. $r = [x,y,z] = [at, bt^2, ct^3]$. $\dot{r} = [a, 2bt, 3ct^2]$ and $dr/ds = \dot{r}/|\dot{r}| = \frac{[a, 2bt, 3ct^3]}{\sqrt{a^2 + 4b^2t^2 + 9c^2t^4}}$

Recall that $A \cdot A = |A|^2$. In particular $\tau \cdot \tau = |\tau|^2 = 1$, but $d/dt(\tau \cdot \tau) =$ $\frac{d\tau}{dt} \cdot \tau + \tau \cdot \frac{d\tau}{dt} = d/dt(1) = 0$. Thus $\tau \cdot \frac{d\tau}{dt} = 0$.

Now the geometric significance of this is that τ must be perpendicular to $\frac{d\tau}{dt}$ (since the product is zero). Thus, we obtain a normal vector to τ. The same argument may be

applied using the parameter s (arc length) instead of t. Thus, we may obtain a normal vector $\frac{d\Upsilon}{ds}$ or $\frac{d\Upsilon}{dt}$ to a particular space curve.



Now if the parameter is s, it can be shown that $\frac{d\Upsilon}{ds}$ measures the rate of change of the direction of Υ and we call the magnitude of this vector the curvature of a curve. The symbol usually used for the curvature is k(kappa). The radius of curvature ρ (rho) = 1/k, i.e., $k = \left|\frac{d\Upsilon}{ds}\right|$ and $\rho = \frac{1}{\left|\frac{d\Upsilon}{ds}\right|}$. This, of course, agrees with the familiar definition of the radius of curvature ds/dϕ given in elementary calculus by the formula $\frac{[1 + (y')^2]^{3/2}}{y''}$ where $\phi = \tan^{-1} dy/dx$.

EXAMPLE 4: Find the radius of curvature of the circle x =acost, y = a sint.

Method 1: r = [acost,asint] and $\dot{r}$ = a[-sint,cost] and dr/ds = [-sint,cost] = Υ.

Now $\frac{d\Upsilon}{ds} = \frac{d\Upsilon}{dt}\cdot\frac{dt}{ds}$, but $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\left|\frac{dr}{dt}\right|}$ $\therefore$ $\frac{d\Upsilon}{ds} = \frac{\frac{d\Upsilon}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{[-\cos t, -\sin t]}{\sqrt{a^2\sin^2 t + a^2\cos^2 t}}$

$= \frac{[\cos t, \sin t]}{-a}$ & $\left|\frac{d\Upsilon}{ds}\right| = \frac{1}{-a}$ $\therefore$ $\rho = -a$

Method 2: $x^2 + y^2 = a^2$. $y' = -x/y$ and $1 + (y')^2 = 1 + x^2/y^2 = a^2/y^2$. Therefore, $[1 + (y')^2]^{3/2} = a^3/y^3$. Now $y'' = \frac{-y + xy'}{y^2} = \frac{-y - x^2/y}{y^2} = -a^2/y^3$. Thus, $\frac{[1 + (y')^2]^{3/2}}{y''} = \frac{a^3/y^3}{-a^2/y^3} = -a$ which checks with method 1.

The unit tangent vector Υ and the normal vector $\frac{d\Upsilon}{ds}$ change in general at each point of the curve. The unit normal vector $\frac{\frac{d\Upsilon}{ds}}{\left|\frac{d\Upsilon}{ds}\right|}$ = N is also of interest, since it conveys certain information about space curves. We may also have another unit vector mutually orthogonal to Υ and N. This unit vector is called the binormal vector B and is obtained by finding Υ x N. The three unit vectors Υ,N,B form a special O-N basis in 3-space called the moving trihedral at each point (x,y,z) on a particular curve. They form a right-handed system and change from point to point along the curve.

It so happens that the plane determined by the vectors Υ and N is called the osculating plane. This plane is important in the further study of space curves. Another scalar quantity which is also important and which tells us the rate at which the curve is twisting out of the osculating plane at some point P, is called the torsion at P - its symbol is T. It is analgous to the curvature at some point P, which as you will recall, was the rate at which the curve was turning away from the tangent line at

P. Knowing the curvature and torsion of curves at any point can tell us a considerable amount about the curves. For example, the following theorems can readily be proven.

<u>Theorem 1</u>: A curve is a straight line iff the curvature is zero.

<u>Theorem 2</u>: A curve which isn't a straight line is a plane curve iff its torsion is zero.

We also have interesting relations between Υ, N, B, k and T. They are:

(1) $d\Upsilon/ds = kN$ (2) $dB/ds = -TN$ (3) $dN/ds = TB - k\Upsilon$.

<u>Example 5</u>: Find the equation of the tangent and normal to the curve $x = t$, $y = t^2$, $z = 2/3t^3$ at $t = 3$. What is the curvature? The torsion?

$r = [t, t^2, 2/3t^3]$. Thus, $\dot{r} = [1, 2t, 2t^2]$ and $|\dot{r}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$.

$$= dr/ds = \dot{r}/|\dot{r}| = \frac{[1,2t,2t^2]}{1+2t^2} = \left[\frac{1}{1+2t^2}, \frac{2t}{1+2t^2}, \frac{2t^2}{1+2t^2}\right]. \text{ Now } \dot{\Upsilon} =$$

$$\left[\frac{-4t}{(1+2t^2)^2}, \frac{(1+2t^2)(2) - 2t(4t)}{(1+2t^2)^2}, \frac{(1+2t^2)(4t) - (2t^2)(4t)}{(1+2t^2)^2}\right]$$

$$= \frac{[-4t, 2-4t^2, 4t]}{(1+2t^2)^2} . |\dot{\Upsilon}| = \frac{2(1+2t^2)}{(1+2t^2)^2} = \frac{2}{(1+2t^2)}. \text{ Therefore } N = \frac{\dot{\Upsilon}}{|\dot{\Upsilon}|} = \frac{[-2t, 1-2t^2, 2t]}{1+2t^2}$$

at $t = 3$, a point on the curve is $(3,9,18)$ and $\Upsilon = [1,6,18]/19$. Thus, the equation of the tangent line is $[3,9,18] + t[1,6,18]$. At $t = 3$, $N = [-6,-17,6]/19$; thus, the equation of the normal line is $[3,9,18] + t[-6,-17,6]$. $d\Upsilon/ds = d\Upsilon/dt \cdot dt/ds = \dfrac{d\Upsilon/dt}{|dr/dt|} = \dfrac{\dot{\Upsilon}}{|\dot{r}|}$

$$= \frac{[-4t, 2-4t^2, 4t]}{(1+2t^3)^3}. \text{ Since } d\Upsilon/ds = kN \text{ and } N = \frac{d\Upsilon/ds}{|d\Upsilon/ds|}, \text{ then } k = |d\Upsilon/ds| = \frac{2}{(1+2t^2)^2}$$

$= 2/361$ at $t = 3$. Now $\Upsilon = [1,6,18]/19$ and $N = [-6,-17,6]/19$, therefore $B = \Upsilon \times N$ $= [18,-6,1]/19$ at $t = 3$.

$$B = \frac{\begin{vmatrix} i & 1 & -2t \\ j & 2t & 1 - 2t^2 \\ k & 2t^2 & 2t \end{vmatrix}}{(1+2t^2)^2} = \frac{[2t^2 + 4t^4, -(2t + 4t^3), 1 + 2t^2]}{(1+2t^2)^2} = \frac{[2t^2, -2t, 1]}{1+2t^2}$$

But $dB/ds = -TN = dB/dt \cdot dt/ds = \dot{B}/|\dot{r}| = -TN$. $\dot{B} = \dfrac{[4t, (-2 + 4t^2), -4t]}{(1+2t^2)^2}$

Therefore $\dot{B}/|\dot{r}| = \dfrac{[4t, -2 + 4t^2, -4t]}{(1+2t^2)^3} = -T\dfrac{[-2t, 1 - 2t^2, 2t]}{(1+2t^2)}$. Thus, $T = \dfrac{2}{(1+2t^2)^2} = 2/361$

<u>EXAMPLE 6</u>: Find Υ, N, B for the circular helix $x = a\cos t$, $y = a\sin t$, $z = at\cot\theta$

$r = [a\cos t, a\sin t, at\cot\theta]$. $\dot{r} = [-a\sin t, a\cos t, a\cot\theta]$ and $|\dot{r}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2\cot^2\theta}$ $= a\csc\theta$. Therefore $\Upsilon = \dot{r}/|\dot{r}| = [-\sin\theta\sin t, \sin\theta\cos t, \cos\theta]$. $\dot{\Upsilon} = [-\sin\theta\cos t, -\sin\theta\sin t, 0]$

$|\dot{\Upsilon}| = \sqrt{\sin^2\theta\cos^2 t + \sin^2\theta\sin^2 t} = \sin\theta$. Now $N = \dfrac{\dot{\Upsilon}}{|\dot{\Upsilon}|} = [-\cos t, -\sin t, 0]$.

$$B = \Upsilon \times N = \begin{bmatrix} i & -\sin\theta\sin t & -\cos t \\ j & \sin\theta\cos t & -\sin t \\ k & \cos\theta & 0 \end{bmatrix} = [\cos\theta\sin t, -\cos\theta\cos t, \sin\theta]$$

EXERCISES(27):

1. Prove the formulas given in article 64.

2. Find the radius of curvature and the torsion of the helix $x = 2\cos t$, $y = 2\sin t$, $z = 2t\tan A$

3. If $x = t^2 - 1$, $y = 2t$, $z = t^2 + 1$, find a unit tangent vector at $t = -1$.

4. If $x = t$, $y = t^2$ in 2-space, find Υ, N at $t = 1$.

5. At what point or points is(are) the tangent(s) to the curve $x = t^3$, $y = 5t^2$, $z = 10t$ perpendicular to the tangent at the point where $t = 1$.

6. Find Υ, N, B for the curve $r(t) = [t, t^2, t^3]$ at $t = 1$.

7. Find Υ, N, B for $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ at $t = \pi$.

8. Find the equation of the osculating plane to $x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$ at $t = 1$.

9. Find the equation of the osculating plane and normal line to $x = 2\sec t$, $y = 2\tan t$, $z = 1$ at $t = 0$.

10. Let $\dot{r} = dr/dt$ and $r' = dr/ds$ where dots are derivatives with respect to t and primes are derivatives with respect to s. Show that $r' = \dot{r}/\dot{s}$, $r'' = \ddot{r}/\dot{s}^2 - \dot{r}(\ddot{s})/\dot{s}^3$.

65. PARTIAL DIFFERENTIATION OF A VECTOR WITH RESPECT TO A SCALAR

Just as a curve in space may be represented by a single parameter t, it can be shown that a surface can be represented by parametric equations involving two parameters u and v. When this is the case, finding partial derivatives with respect to u and v prove useful. It can readily be seen that with the usual definitions of continuity, differentiability and the like, we may obtain partial derivatives of vectors by the usual means i.e., let one of the parameters remain constant and find the derivative with respect to the other. Thus, $\frac{\partial}{\partial u}(A\cdot B) = A\cdot\frac{\partial B}{\partial u} + \frac{\partial A}{\partial u}\cdot B$, $\frac{\partial}{\partial u}(A \times B) = \frac{\partial A}{\partial u} \times B + A \times \frac{\partial B}{\partial u}$ etc. as in the single variable formulas above. In particular, when we are given the equation of a surface involving the parameters u and v, then $\frac{\partial r}{\partial u}$ will represent a vector which is tangent to the curve where v is constant and $\frac{\partial r}{\partial v}$ will represent a vector which is tangent to the curve where u is constant. If we find $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$, we must therefore have a normal to the surface at any point P.

EXAMPLE 1: Find the normal to the surface $z = x^2 + y^2$ at the point $(-1,2,2)$.

First, we put the equation of the surface in parametric form. We can let $x = u$, $y = v$ and $z = u^2 + v^2$. Thus, $r = [u, v, u^2 + v^2]$ and $\frac{\partial r}{\partial u} = [1,0,2u]$; $\frac{\partial r}{\partial v} = [0,1,2v]$.

Now $u = x = -1$; $v = y = 2$. Therefore, $\frac{\partial r}{\partial u} = [1,0,-2]$ $\frac{\partial r}{\partial v} = [0,1,4]$.

The normal vector $= \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{bmatrix} i & 1 & 0 \\ j & 0 & 1 \\ k & -2 & 4 \end{bmatrix} = [2,-4,1]$

Note that the total differential $dr = \frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv$ where r depends on the parameters u and v. Recall also that $dr = [dx,dy,dz]$ in 3-space and thus $dr \cdot dr$ would become:

$[dx,dy,dz] \cdot [dx,dy,dz] = dx^2 + dy^2 + dz^2 = ds^2$.

<u>EXAMPLE 2</u>: Find ds^2 in terms of partial derivatives in u,v.

We have $dr = \frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv$ and thus, $dr \cdot dr = (\frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv) \cdot (\frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv)$

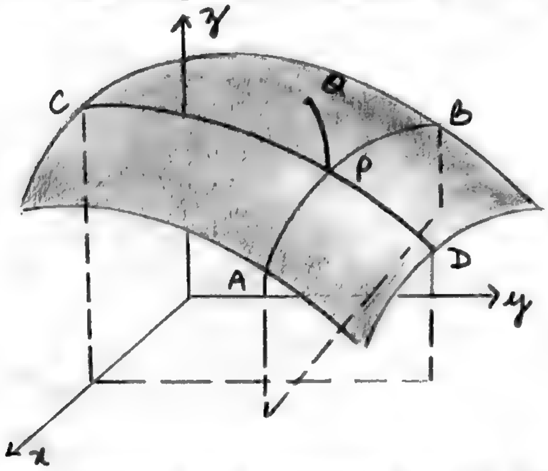
$= \left(\frac{\partial r}{\partial u}\right)^2 du^2 + 2\frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial v}dudv + \left(\frac{\partial r}{\partial v}\right)^2 dv^2 = ds^2$.

<u>EXERCISES(28)</u>:

1. Find a normal vector to the surface $z = xy$.
2. Find the equation of the tangent plane and the normal line to the surface $z = xy$ at the point $(2,5,10)$.
3. Find the equation of the tangent plane to $2z = x^2 - y^2$ at $(1,2,3)$.
4. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 4$ & $(x - 2)^2 + y^2 + z^2 = 9$.
5. Find the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_1,y_1,z_1).

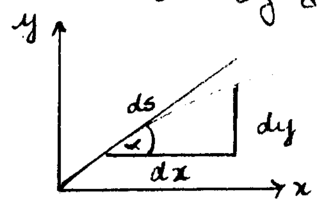
66. <u>DIFFERENTIATION OF A SCALAR WITH RESPECT TO A VECTOR</u>

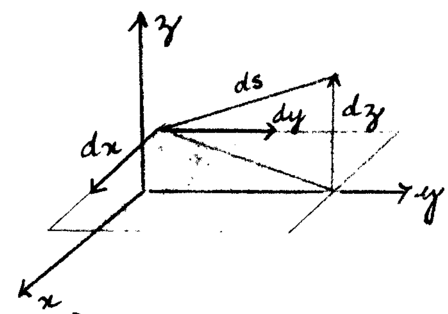
Recall from the elementary calculus that $z = f(x,y)$ represented a surface and that $\frac{\partial z}{\partial x}$ represented the slope at any point of the tangent line determined by a plane parallel to the xz-plane cutting the surface(y being constant). Similarly, $\frac{\partial z}{\partial y}$ represented the slope of the tangent line obtained by cutting the surface with a plane parallel to the yz-plane(x remaining constant). In the figure below, the two curves that one finds tangents to are designated by AB and CD respectively when taking partials. $\frac{\partial z}{\partial x}$ = slope of APB at P(any point (x,y,z)).(keep y constant). $\frac{\partial z}{\partial y}$ = slope of CPD at P. (keep x constant) We may also want the slope of PQ where PQ might be any cock-eyed curve obtained by cutting the surface with a plane at an oblique angle. It can be shown that this slope is given by dz/ds where s is the parameter of arc length. The derivative dz/ds is called the <u>directional derivative</u>(in a given direction). As a practical example,



one might consider himself in a balloon measuring temperatures at certain points - say at point A, the temperature would be 3° and at point B, the temperature would be 6°. He would then estimate that the directional derivative of the temperature(in the direction $\overrightarrow{AB}$) would be 3° per whatever distance he has travelled between points A and B. If he continues in the same direction, he would assume that the temperature would rise accordingly. If the balloon moves on a curved path at a constant speed, the balloonist can calculate the directional derivative along the path without looking out of the balloon. Formally, from the elementary calculus, we have:

$dz/ds = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$ but we recall that $dx/ds = \cos\alpha$ and $dy/ds = \sin\alpha = \cos\beta$





$(\beta = 90 - \alpha)$. Thus, in 2-space $dz/ds = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\cos\beta$(refer to the 1st figure above). In 3-space, we extend this concept somewhat and referring to the second illustration above, we immediately realize that $dx/ds = \cos\alpha$, $dy/ds = \cos\beta$, $dz/ds = \cos\gamma$. The above concepts may be extended by means of the total differential. Since when $f(x,y,z) = f$, then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$; this suggests that we might define df as the product of 2 vectors $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]\cdot[dx,dy,dz]$ and further we could "detach" $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$ using it as an "operator vector" which would operate on a scalar f. We call this operator vector ∇ ("del" or "nabla") and thus, we have $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \nabla f$ where f is some scalar. This can be looked upon as differentiating a scalar with respect to a vector. From the above then, $df = \nabla f\cdot dr$(since $dr = [dx,dy,dz]$). Note that df is a scalar but ∇f is a vector! In particular, if $f(x,y,z) = C$(a surface in 3-space), then $df = 0$, i.e., $\nabla f\cdot dr = 0$..Now dr is some vector lying in the tangent plane at the point (x,y,z) by definition and therefore ∇f must be a vector perpendicular to a vector in the tangent plane, i.e. ∇f is a vector normal to the surface $f(x,y,z) = C$. This particular normal vector ∇f is called grad f(the gradient of f)

EXAMPLE 1: Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at $(-1,2,2)$. First we write $z = x^2 + y^2$ in the form $f(x,y,z) = C$, i.e. $f = x^2 + y^2 - z = 0$.

Now $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [2x, 2y, -1]$ which is a normal vector at any point (x,y,z) on the surface. At the particular point $(-1,2,2)$, we have $[-2,4,-1]$. (Compare with example 1, section 65.). We take our universal instance (x,y,z) in the plane and we have a vector $[x + 1, y - 2, z - 2]$ as some vector in the required plane. But we know that $[x + 1, y - 2, z - 2] \cdot [-2,4,-1] = 0$ since the vectors are perpendicular. Therefore the required equation is: $-2x + 4y - z - 8 = 0$.

The directional derivative in 3-space can be neatly expressed as $\nabla f \cdot dr/ds = \nabla f \cdot \Upsilon$ This is quickly derived from $df/ds = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$

<u>EXAMPLE 2</u>: Find the directional derivative of $\emptyset = xyz + 2xz^2$ at $(-1,2,2)$ in the direction $[2,1,-2]$.

Clearly, $\nabla\emptyset = [yz + 2z^2, xz, xy + 4xz] = [12,-2,-10]$ at $(-1,2,2)$. Now dr/dt is already given as $[2,1,-2]$ so $\Upsilon = [2,1,-2]/3$. Therefore $df/ds = [12,-2,-10] \cdot [2,1,-2]/3 = 14$.

<u>EXERCISES(29)</u>:

1. Verify the results obtained in exercises(28) above by using the gradient.

2. Evaluate the directional derivative of the following functions for the points and directions given: (a) $2x^2 - y^2 + z^2$ at $(1,2,3)$ in the direction of the line from $(1,2,3)$ to $(3,5,0)$. (b) $e^x\cos y$ at $(0,0)$ in a direction making an angle of 60° with the x-axis. (c) $2x - 3y$ at $(1,1)$ along the curve $y = x^2$ in the direction of increasing x. (d) $3x - 5y + 2z$ at $(2,2,1)$ in the direction of the curve $x^2 + y^2 - z^2 = 0$ and $2x^2 + 2y^2 - z^2 = 25$ in the direction of increasing x.

3. Find the tangent plane and normal lines to the surfaces at the points indicated: (a) $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$. (b) $e^{x^2+y^2} - z^2 = 0$ at $(0,0,1)$. (c) $x^3 - xy^2 + y^2 - z^3 = 0$ at $(1,1,1)$. (d) $z = x^2 + y^2$ at $(1,1,2)$. (e) $z = \sqrt{1 - x^2 - y^2}$ at $(2/3,2/3,1/3)$.

4. Determine a plane normal to the curve $x = t^2$, $y = t$, $z = 2t$ and passing through $(1,0,0)$.

5. Show that the curve $x^2 - y^2 + z^2 = 1$, $xy + xz = 2$ is tangent to the surface: $xyz - x^2 - 6y = -6$ at $(1,1,1)$

67. <u>THE DIVERGENCE AND CURL OF A VECTOR</u>

From the above discussion it was seen that we could consider the operator ∇ as a vector $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$ and further that we could use this operator to change a scalar function f into a vector. It would be natural to inquire whether we can form the dot amd cross-products of vectors with ∇ and further, to see if there would be any physical significance or meaning to such operations. Formally then $\nabla \cdot A$ should turn out to be a scalar quantity and would yield the following in 3-space, for example:

$$\nabla \cdot A = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [A_1, A_2, A_3] = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}.$$

This entity is known as the divergence of a vector. It can be shown that the physical significance of $\nabla\cdot\vec{A}$ is that it gives the rate per unit volume for each point at which the physical entity is issuing from that point. For a particular example, suppose A is the velocity of air escaping from an automobile tire during a blow-out; then div A would be the volume of air/unit time/unit volume escaping from a hole surrounding the point. If after repairs, air is pumped back into the tire, the divergence of the velocity of air inside the tire would be negative.

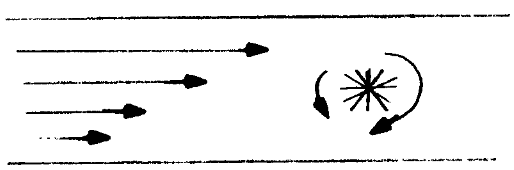
Perhaps at this point, it would be opportune to discuss the concept of a "field". If in any region of space a function f is defined for each point in that region then this region is called a field. For our discussions we must always consider that the functions that we deal with are continuous, single-valued and contain continuous first derivatives in the aforementioned regions. Now if f is defined in terms of quantities (x,y,z) for instance where f(x,y,z) is a scalar, then this field is called a scalar field. Similarly, if f(x,y,z) is a vector then the field is a vector field. Some physical examples of the former field are temperature and electrostatic potential; of the latter field are fluid flow and force fields arising from electromagnetic sources and gravitation. It is seen that scalar fields give rise to vector fields (e.g. the field of the vector ∇f) and vector fields give rise to scalar fields (e.g. the field of $\nabla\cdot\vec{A}$).

Now if some physical entity is generated within a certain region of a field, that region is termed a source. On the other hand if the physical entity is absorbed, then the region is called a sink. If the strengths of the sources are greater than that of the sinks, the net outflow is said to be positive and conversely. It can be shown that if $\nabla\cdot A = 0$ in a region then there are no sources or sinks and vv. Of course, if $\nabla\cdot A$ is positive at some point P, then there must be a source located at P - for example, if A represents the flow of heat and $\nabla\cdot A$ is positive then there is either a source of heat at P or else heat must be leaving P so that the temperature is decreasing.

The cross-product of with any vector A is called the curl of A and is defined in the usual manner, i.e.

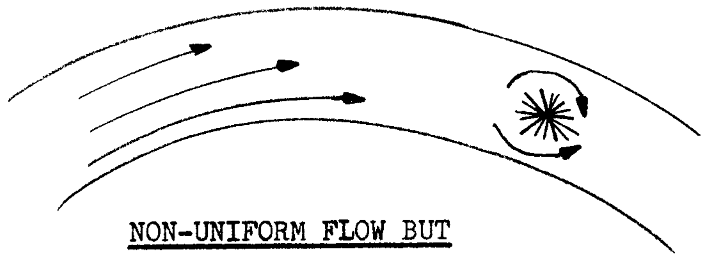
$$\nabla\times A = \begin{bmatrix} i & \frac{\partial}{\partial x} & A_1 \\ j & \frac{\partial}{\partial y} & A_2 \\ k & \frac{\partial}{\partial z} & A_3 \end{bmatrix} = \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right].$$

The physical significance of the curl of a vector for A gives a measure for the angular velocity at every point of the vector field. In the example where we discussed the blowout above, $\nabla \times A$ can be described as two times the angular velocity of the air(when the hole is round). The direction of $\nabla \times A$ is normal to the area which is along the axis of rotation(as might be expected being a cross-product). Another way of viewing the curl would be to pretend that we have a small light paddlewheel with many spokes placed in a turbulent stream. Then it will rotate with an angular velocity proportional to the magnitude of the curl of the stream velocity and the axis of rotation of this paddlewheel will be in the direction of the curl. This is illustrated pictorially below:



NON-UNIFORM FLOW

THUS, CURL EXISTS!



NON-UNIFORM FLOW BUT

CURL = 0!

Needless to say, there are a variety of applications in the study of heat, fluid dynamics and electromagnetism, but we will confine our interests mostly to geometry. Triple products can also be found and there are two important theorems about the curl and divergence. The proofs will not be given.

Theorem 1: The curl of a vector A is zero iff $A = \nabla f$.(i.e. $\nabla \times A = 0$ iff $A = \nabla f$).

Theorem 2: The divergence of a vector A is zero iff $A = \nabla \times B$(i.e. $\nabla \cdot A = 0$ iff $A = \nabla \times B$).

EXAMPLE 1: Show that $\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$.

l.h.s.: $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [A_1 + B_1, A_2 + B_2, A_3 + B_3] = \frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial B_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_3}{\partial z} =$

$\nabla \cdot A + \nabla \cdot B$ QED

EXAMPLE 2: Show that $\nabla \times (\phi A) = (\nabla \phi) \times A + \phi(\nabla \times A)$.

l.h.s.: $= \begin{bmatrix} i & \frac{\partial}{\partial x} & \phi A_1 \\ j & \frac{\partial}{\partial y} & \phi A_2 \\ k & \frac{\partial}{\partial z} & \phi A_3 \end{bmatrix} = \left[\frac{\partial(\phi A_3)}{\partial y} - \frac{\partial(\phi A_2)}{\partial z}, \frac{\partial(\phi A_1)}{\partial z} - \frac{\partial(\phi A_3)}{\partial x}, \frac{\partial(\phi A_2)}{\partial x} - \frac{\partial(\phi A_1)}{\partial y}\right]$

but $\frac{\partial(\phi A_3)}{\partial y} = \frac{\partial \phi}{\partial y} A_3 + \frac{\partial A_3}{\partial y}\phi$ etc. Therefore we have, $\left[\frac{\partial \phi}{\partial y} A_3 + \frac{\partial A_3}{\partial y}\phi - \frac{\partial \phi}{\partial z} A_2 - \frac{\partial A_2}{\partial z}\phi, \frac{\partial \phi}{\partial z} A_1 + \frac{\partial A_1}{\partial z}\phi - \frac{\partial \phi}{\partial x} A_3 - \frac{\partial A_3}{\partial x}\phi, \frac{\partial \phi}{\partial x} A_2 + \frac{\partial A_2}{\partial x}\phi - \frac{\partial \phi}{\partial y} A_1 - \frac{\partial A_1}{\partial y}\phi\right] = (\nabla \phi) \times A + \phi(\nabla \times A)$.

Working out other formulas involving div and curl are fairly strightforward but rather tedious. A list of the more useful formulas is shown below:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla\cdot(A+B) = \nabla\cdot A + \nabla\cdot B$
3. $\nabla \times (A+B) = \nabla \times A + \nabla \times B$
4. $\nabla\cdot(\phi A) = (\nabla\phi)\cdot A + \phi(\nabla\cdot A)$
5. $\nabla \times (\phi A) = (\nabla\phi)\times A + \phi(\nabla \times A)$
6. $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$
7. $\nabla\times(A\times B) = (B\cdot\nabla)A - B(\nabla\cdot A) - (A\cdot\nabla)B + A(\nabla\cdot B)$
8. $\nabla(A\cdot B) = (B\cdot\nabla)A + (A\cdot\nabla)B + B\times(\nabla\times A) + A\times(\nabla\times B)$
9. $\nabla\cdot(\nabla\phi) = \nabla^2\phi$
10. $\nabla\times(\nabla\phi) = 0 = \nabla\cdot(\nabla\times A)$ (thms 1 and 2).
11. $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

NOTE: $(A\cdot\nabla)B$ IS A VECTOR!
$(A\cdot\nabla)B = A\cdot\nabla B = [A\cdot\nabla B_1, A\cdot\nabla B_2, A\cdot\nabla B_3]$

EXAMPLE 3: Prove that $\nabla\cdot(\vec{r}/r^3) = 0$.

Let $\vec{r} = A$ and $r^{-3} = \phi$. Then we have $\nabla\cdot(\phi A)$. By (3) above we have,
$\nabla\cdot(\phi A) = (\nabla\phi)\cdot A + \phi(\nabla\cdot A)$. Now $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ and $r^{-3} = (x^2+y^2+z^2)^{-3/2}$.
Thus, $\nabla(r^{-3}) = -3/2(x^2+y^2+z^2)^{-5/2}[2x,2y,2z]$, i.e. $\nabla(r^{-3}) = -3r^{-5}[x,y,z] = \nabla\phi$;
but $\nabla\phi\cdot A = -3r^{-5}[x,y,z]\cdot[x,y,z]$. Therefore, $\nabla\phi\cdot A = -3r^{-5}(x^2+y^2+z^2) = -3r^{-3}$.
Now $\nabla\cdot A = \nabla\cdot r = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ and $\phi(\nabla\cdot r) = 3r^{-3}$. Therefore we have,
$\nabla\cdot(\vec{r}/r^3) = -3r^{-3} + 3r^{-3} = 0$.

EXAMPLE 4: Find $\nabla\times A$ at $(1,-1,0)$ if $A = [2x,2xz,3yz^3]$.

$$\nabla\times A = \begin{bmatrix} i & \frac{\partial}{\partial x} & 2x \\ j & \frac{\partial}{\partial y} & 2xz \\ k & \frac{\partial}{\partial z} & 3yz^3 \end{bmatrix} = [3z^3 - 2x, 0, 2z] \text{ and at } (1,-1,0) = [-2,0,0].$$

EXERCISES(30):

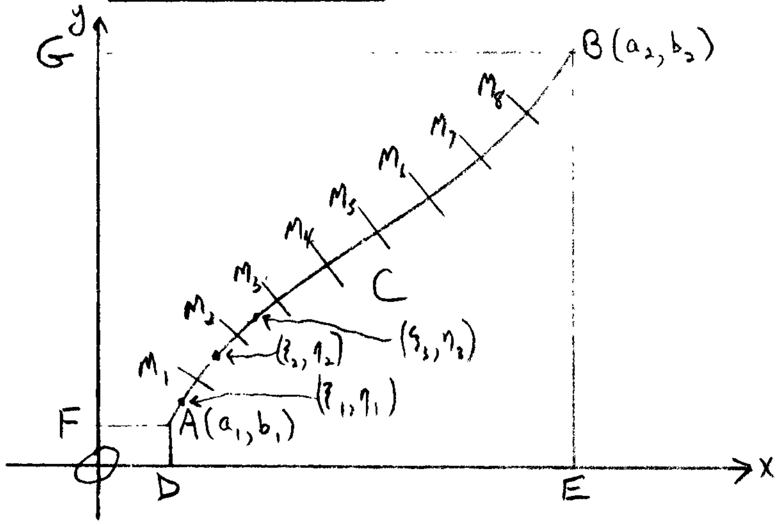
1. Prove the formulas above given in article 67.
2. Show that $\nabla\cdot\nabla f = \nabla^2 f$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
3. (a) if $f(x,y,z) = x^2y - y^2z + x^2y\,z^2$, find curl(grad f).
   (b) If $f(x,y,z) = xy + y^2z + x^2yz$, find div(curl f).
4. Find div A if (a) $A = [x,y,z]$; (b) $A = [z-y, x-z, y-x]$.
5. Show that $\nabla\times(\nabla\times A) = -\nabla^2 A + \nabla(\nabla\cdot A)$ where ∇^2 is defined as in problem 2.

CHAPTER 13 - VECTOR INTEGRATION

This chapter will deal with vector integration in a rather cursory fashion but with the ultimate goal of establishing both the divergence and Stokes' theorems. To do this, we must go through a few preliminaries and again the student must be reminded that an intuitive, non-rigorous but pragmatic approach will be taken. Rigorous discussions and proofs of any of the theorems below can be found in any good calculus text so that if there are any further mysteries regarding the problems presented, the

student should peruse these texts.

68. THE LINE INTEGRAL



Consider a function P(x,y) defined and continuous for a certain region of the Plane OXY and take a curve C in this region extending from point A to point B. Now divide this curve C into n segments by the points $M_1(x_1,y_1)$, $M_2(x_2,y_2)$..... etc. Let (ξ_i, η_i) be a point in the segment (M_{i-1},M_i) and form the sum,

$P(\xi_1, \eta_1)(x_1 - x_0) + P(\xi_2, \eta_2)(x_2 - x_1) + \ldots\ldots\ldots + P(\xi_n, \eta_n)(x_n - x_{n-1})$. In the short form, we would have, $\sum_{i=1}^{n} P(\xi_i, \eta_i)(x_i - x_{i-1})$. The limit of this sum as n approaches infinity and as $|x_i - x_{i-1}|$ approaches zero is the line integral of P(x,y) along curve C and is symbolized by $\int_{(a_1,b_1)}^{(a_2,b_2)} P(x,y)dx$. Note that the value of this integral depends not only on its limits but also on the particular curve over which the sum is found.

EXAMPLE 1: Evaluate $\int_{(0,0)}^{(1,2)} (x + y^2)dx$ along: (a) a straight line y = 1 (b) a parabola $y^2 = 4x$.

(a) if y = 1 then $y^2 = 1$ and we put the line integral into the usual form using the appropriate limits(in this case the x values) and we have, $\int_0^1 (x + 1)dx = x^2/2 + x\Big]_0^1 = 3/2$.

(b) If $y^2 = 4x$ then, the line integral becomes: $\int_0^1 (x + 4x)dx = \int_0^1 5xdx = 5x^2/2\Big]_0^1 = 5/2$.

Note that if P(x,y) happened to be y, we would have $\int_{(a_1,b_1)}^{(a_2,b_2)} P(x,y)dx = \int_{a_1}^{a_2} ydx =$ area of ADEB where C is the upper boundary and if $P(x,y) = \pi y^2$, then the line integral $\int_L$ is $\pi\int_{a_1}^{a_2} y^2dx$ which is the volume of the solid formed by revolving ADEB around OX. Thus, the line integral becomes a rather mysterious entity - an integral which changes into different forms i.e., other types of integrals which depend on the way we define the particular curve C.

Now if Q(x,y) is yet another function, we may form another sum $\sum_{i=1}^{n} Q(\xi_i, \eta_i)(y_i - y_{i-1})$ and find the limit as n approaches infinity and $|y_i - y_{i-1}|$ approaches zero. This will yield another line integral which we designate by $\int_{(a_1,b_1)}^{(a_2,b_2)} Q(x,y)dy$ taken along the curve C.

Notice that it follows that if Q(x,y) = x, for instance, then $\int_L = \int_{b_1}^{b_2} xdy$ = area FABG(of which C is the boundary) and if $Q(x,y) = \pi x^2$ then $\int_L$ is the volume of the solid formed by revolving FABG about OY!

In practice it is more common to find $\int_L$ written as $\int_{(a_1,b_1)}^{(a_2,b_2)} [P(x,y)dx + Q(x,y)dy]$ and

for brevity, the form in 2-space suggests a vector form $\int_C A\cdot dr$ where $P(x,y) = A_1$; $Q(x,y) = A_2$ and $[dx,dy] = dr$. In 3-space, of course, we would have $A_1 = P(x,y,z)$; $A_2 = Q(x,y,z)$; $A_3 = R(x,y,z)$ and the line integral $\int A\cdot dr$ would take on the form: $\int_C Pdx + Qdy + Rdz$ and would be defined along the same curve in space in a manner precisely similar to the definition of a line integral along a plane curve.

EXAMPLE 2: Evaluate $\int_{(0,0)}^{(1,2)} (x + y^2)dx + 2xy^2dy$ along: (a) $y = 1$; (b) $y^2 = 4x$.

(a) If $y = 1$ then $dy = 0\cdot dx = 0$. Therefore $\int_0^1 (x + 1)dx = x^2/2 + x\Big]_0^1 = 3/2$.

(b) $y^2 = 4x$ implies that $2ydy = 4dx$, thus $\int_L = \int_0^2 (y^2/4 + y^2)2ydy/4 + 2(y^2/4)y^2dy$ $= \frac{1}{2}\int_0^2 (5y^3/4 + y^4)dy = \frac{1}{2}\Big[5y^4/6 + y^5/5\Big]_0^2 = \frac{1}{2}\Big[5 + 32/5\Big] = 5/2 + 32/10 = 57/10.$

EXAMPLE 3: $\int_{(0,0,0)}^{(1,1,1)} y^2dx + xydy + xzdz$ along (a) a straight line from the upper limit to the lower limit; (b) $x = t$, $y = t^2$, $z = t^3$.

(a) The equation of the straight line from $(0,0,0)$ to $(1,1,1)$ is $x/1 = y/1 = z/1$ or $[0,0,0] + t[1,1,1]$. Therefore $x = t$, $y = t$, $z = t$ and $dx = dt = dy = dz$. When $x = 0$, $t = 0$; when $x = 1$, $t = 1$. Changing to parameter t, we have:

$\int_0^1 t^2dt + t^2dt + t^2dt = \int_0^1 3t^2dt = t^3\Big]_0^1 = 1.$

(b) If $x = t$, $dx = dt$; if $y = t^2$, $dy = 2tdt$; if $z = t^3$, $dz = 3t^2dt$. Now when $x = y = z = 0$, then $t = 0$; when $x = 1$, $t = 1$; $y = 1$, $t = 1$; $z = 1$, $t = 1$. Therefore $\int_L$ becomes:

$\int_0^1 t^4dt + t^3\cdot 2tdt + t^4\cdot 3t^2dt = \int_0^1(3t^4 + 3t^6)dt = 3\Big[t^5/5 + t^7/7\Big]_0^1 = 36/35.$

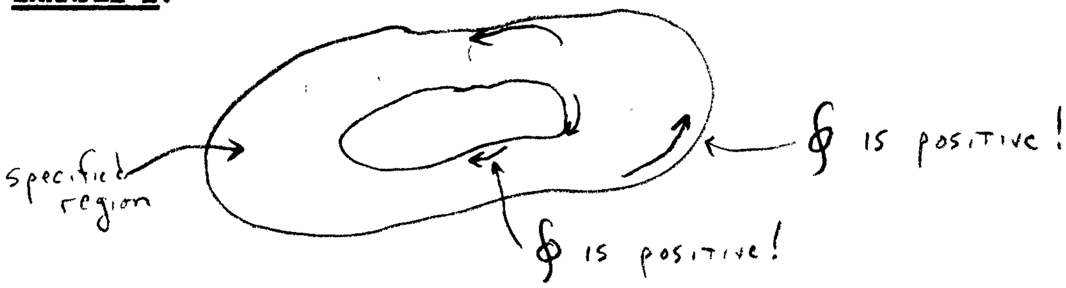
EXERCISES(31):

1. Find the value of $\int_{(0,0)}^{(1,2)} (x + y^2)dx + 2xy^2dy$ along: (a) a straight line (b) $y^2 = 4x$.
2. Evaluate $\int_{(0,0)}^{(\pi/2,\pi/2)}$ $y\cos x dx + \sin x dy$ along (a) $y = x$; (b) $y^2 = 4x$.
3. Evaluate $\int_{(0,0)}^{(1,1)} (x^2 + y^2)dx - 2xydy$ along (a) $y = x$; (b) $x = y^2$.
4. Evaluate $\int_{(1,0,0)}^{(1,0,2\pi)} zdx + xdy + ydz$ along $x = \cos t$, $y = \sin t$, $z = t$.
5. Evaluate $\int_{(1,0,1)}^{(2,3,2)} x^2dx - xzdy + y^2dz$ along the straight line joining the 2 points.

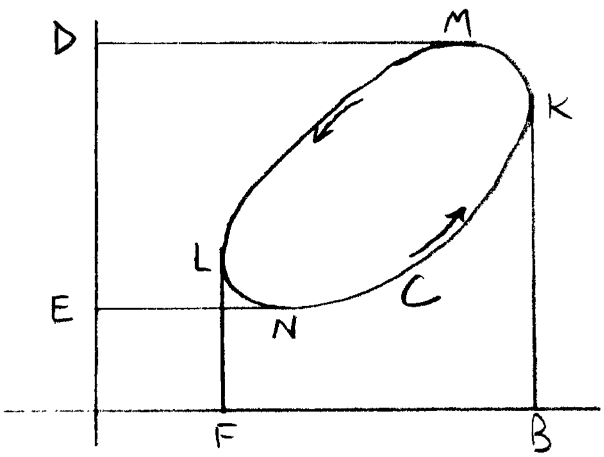
69. LINE INTEGRALS AROUND CLOSED CURVES

To use line integrals about closed curves where the curve is the boundary of some specified region, we need a convention to distinguish the direction. The convention adopted is that if a person can walk around a curve with the region in question on his left-hand side, then the line integral is considered positive and the region in question is also positive. The symbol for the line integral about a closed curve is $\oint_C$.

EXAMPLE 1:



Now consider the following region:

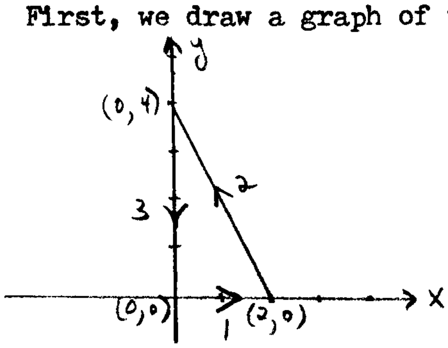


We wish to find area LNKM which would be $\oint_C ydx$. Area LNKM = area FLMKB - area FLNKB and since we wish no confusion with signs we will traverse in such a manner that we retain the usual limits of integration about the curve C. The arrows indicate the postive sense in the figure.

area FLMKB = $-\int_{BKMLF} ydx$ (larger area) since we have region on right when walking along C. Area FLNKB = $\int_{FLNKB} ydx$(smaller area) since we have region on left when walking along C. Net result is that Area LNKM = area FLMKB - area FLNKB $= -\oint ydx = A$. Also area DMLNE = $-\int_{ENLMD} xdy$(smaller area) and area ENKMD = $\int_{ENKMD} xdy$(larger area). Net result is that area LNKM = area ENKMD - area DMLNE = $\oint_C xdy = A$. Therefore, we have, $2A = \oint xdy - \oint ydx$ and thus $A = \frac{1}{2}\oint(xdy - ydx)$.

EXAMPLE 2: Find the area of the triangle in the plane determined by (0,0),(2,0),(0,4) by use of line integrals.

First, we draw a graph of the triangle and we have:



Next we zip along the line in the direction determined by (0,0) and (2,0). This is indicated by line integral 1. Subsequently, we zip along the line from (2,0) to (0,4) and finally along the y-axis from (0,4) to (0,0). The sums of line integrals 1 + 2 + 3 will give us the required area. Thus, the area of the triangle is given by:

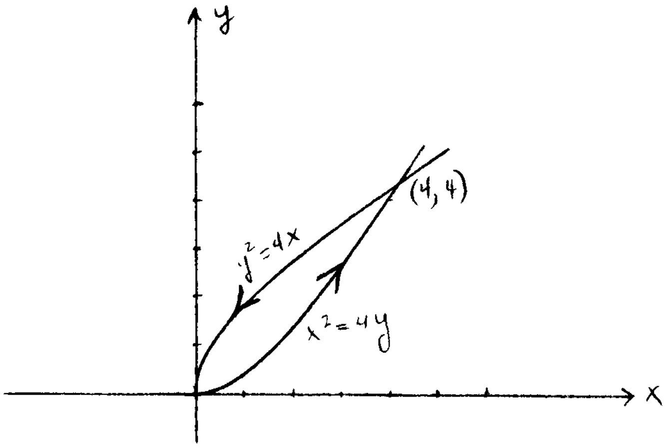
$$\frac{1}{2}\int_{(0,0)}^{(2,0)}(xdy - ydx) + \frac{1}{2}\int_{(2,0)}^{(0,4)}(xdy - ydx) + \frac{1}{2}\int_{(0,4)}^{(0,0)}(xdy - ydx).$$

Along path 1: $y = 0$; $dy = 0$ and $\frac{1}{2}\int_{(0,0)}^{(2,0)}(xdy - ydx) = 0$.

Along path 2: $y = -2x + 4$; $dy = -2dx$. Thus $\frac{1}{2}\int_{(2,0)}^{(0,4)}(xdy - ydx) = \frac{1}{2}\int_2^0 x(-2dx) - (2x + 4)dx = 4$.

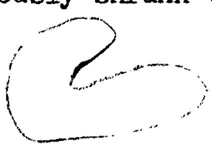
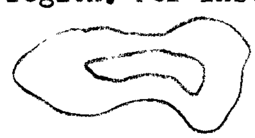
Along path 3: $x = 0$; $dx = 0$ and $\frac{1}{2}\int_{(0,4)}^{(0,0)}(xdy - ydx) = 0$. Thus the area is 4.

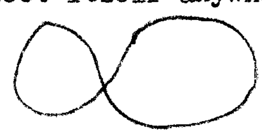
EXAMPLE 3: Find the area between $x^2 = 4y$ and $y^2 = 4x$ by use of line integrals.



Referring to the drawing at the left, we see that we can zip along $x^2 = 4y$, then down along $y^2 = 4x$. Therefore, we have $\frac{1}{2}\int_{(0,0)}^{(4,4)} xdy - ydx$ for the 1st integral and $\frac{1}{2}\int_{(4,4)}^{(0,0)} xdy - ydx$ for the 2nd integral. The first integral becomes: $\frac{1}{2}\int_0^4 (x\cdot x/2 - x^2/4)dx$, since $x^2 = 4y$, then $dy = x/2dx$ and the 2nd integral becomes: $\frac{1}{2}\int_4^0 (y^2/4 - y\cdot y/2)dy$ since $y^2 = 4x$ and $dx = y/2dy$. Therefore we have, $\oint = \frac{1}{2}\int_0^4 x^2/4dx - \frac{1}{2}\int_4^0 y^2/4dy = \frac{1}{2}\Big[x^3/24\Big]_0^4 - \frac{1}{2}\Big[y^3/24\Big]_4^0 = 16/3$.

70. GREEN'S THEOREM IN THE PLANE

It can be shown that if $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$, are single-valued, continuous in a simply connected region bounded by a simple closed curve, then we have the following relation: $\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$. Recall that a single-valued function means that we have one value only of y for a particular value of x or in 3-space, we have 1 value only of z for f(x,y). A simply connected curve means that any closed curve in a region can be continuously shrunk to a point without leaving the region. For instance, the closed curve  is simply connected while  is not!

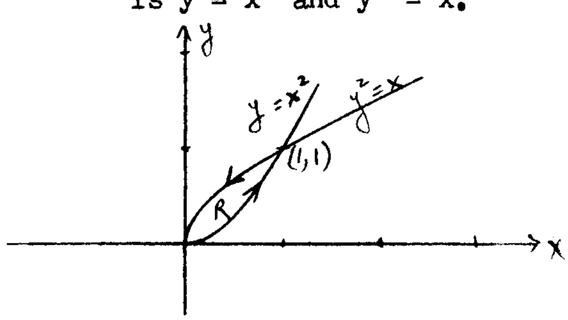
A simple closed curve is a closed curve which does not intersect itself anywhere. For instance, the curve  is simply closed while  is not!

EXAMPLE 1: Verify Green's theorem for the area between $y^2 = 4x$ and $x^2 = 4y$.

Now $A = \frac{1}{2}\oint(xdy - ydx)$. Therefore $P = -y/2$; $Q = x/2$. Thus, $\frac{\partial P}{\partial y} = -\frac{1}{2}$, $\frac{\partial Q}{\partial x} = \frac{1}{2}$. We have $\iint_R dxdy$ as expected. If we integrate with respect to x first, then we have:

$$\int_0^4 \int_{x=2\sqrt{y}}^{y^2/4} dx\,dy = \int_0^4 (y^2/4 - 2\sqrt{y})dy = y^3/12 - 4y^{3/2}/3\Big]_0^4 = 16/3$$ which checks with the previous result given in example 3, section 69 above.

EXAMPLE 2: Verify Green's theorem in the plane for: $\oint(2xy - x^2)dx + (x + y^2)dy$ where C is $y = x^2$ and $y^2 = x$.



$\oint_C = \int_{(0,0)}^{(1,1)} + \int_{(1,1)}^{(0,0)}$ When $y = x^2$, $dy = 2xdx$

$\int_{(0,0)}^{(1,1)} = \int_0^1 (2x^3 - x^2)dx + (x + x^4)2xdx = 7/6$.

When $x = y^2$, $dx = 2ydy$. Thus, $\int_{(1,1)}^{(0,0)} =$

$\int_1^0 (2y^3 - y^4)2ydy + 2y^2dy = -17/15$

Final result is $7/6 - 17/15 = 1/30$.

Now $\iint_R (Q_x - P_y)dxdy$ (using the subscript notation for partials) will be:

$P = 2xy - x^2$; $Q = x + y^2$. Thus, $P_y = 2x$; $Q_x = 1$. The above integral becomes:

$$\int_0^1 \int_{x=y^2}^{x=\sqrt{y}} (1 - 2x)dxdy = \int_0^1 (x - x^2) \quad dy = \int_0^1 (-y^2 + y^4 + y^{\frac{1}{2}} - y)dy = 1/30.$$

71. INDEPENDENCE OF PATH

It can be shown that in 2-space if $Q_x = P_y$, a line integral's value is completely independent of the path one uses to evaluate it. Thus, it isimmediately seen that the line integral about any simple closed curve always has to be zero under these conditions, since zipping from point A to point B will give a value k,say, and then zipping back from B to A will have to give a value -k(if the paths are independent) so that the resulting value has to be zero. The theorem holds true in 3-space as well, but 3 conditions have to be met - namely: $Q_x = P_y$; $R_x = P_z$; $R_y = Q_z$.

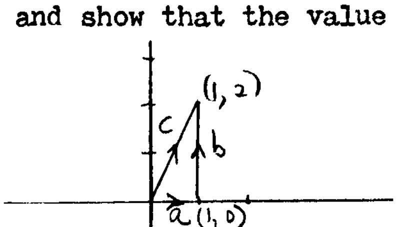
Recall that the vector form of the line integral was $\int_C A \cdot dr$ where $A = [P,Q,R]$ and $dr = [dx,dy,dz]$ in 3-space. It will be shown later(as a consequence of Stokes' theorem) that the condition for independence of path is that $\nabla \times A = 0$. This can be verified, for in 2-space we have:

$$\begin{bmatrix} i & \frac{\partial}{\partial x} & P \\ j & \frac{\partial}{\partial y} & Q \\ k & 0 & 0 \end{bmatrix} = [0,0,Q_x - P_y] = 0.$$

Therefore, we get, $Q_x = P_y$. In 3-space the other conditions also are verified.

EXAMPLE 1: Show that $\int_{(0,0)}^{(1,2)} [3x(x + 2y)dx + (3x^2 - y^3)dy]$ is independent of the path and find its value.

$P = 3x(x + 2y)$ and $P_y = 6x$; $Q = (3y^2 - y^3)$ and $Q_x = 6x$. Hence we may choose any path to evaluate this integral. For verification in this example, we will choose 2 paths and show that the value is the same. First, we will zip along a, then b to get to (1,2)



Then we will zip along c. Along a and b, we have:

Path a: $y = 0$, $dy = 0$; Path b: $x = 1$, $dx = 0$.

$\int_0^1 3x^2dx$ $\qquad$ $\int_0^2 (3 - y^3)dy$

Path a + path b = $\int_0^1 3x^2dx + \int_0^2 (3 - y^3)dy = 3$. Path c: $y = 2x$, $dy = 2dx$. This gives:

$\int_0^1 3x(x + 4x)dx + (3x^2 - 8x^3)2dx = \int_0^1 (15x^2 + 6x^2 - 16x^3)dx = 3.$

EXAMPLE 2: Show that $\int_{(0,1,0)}^{(3,2,-1)} z^2dx + 2ydy + 2xzdz$ is independent of the path of integration and find its value.

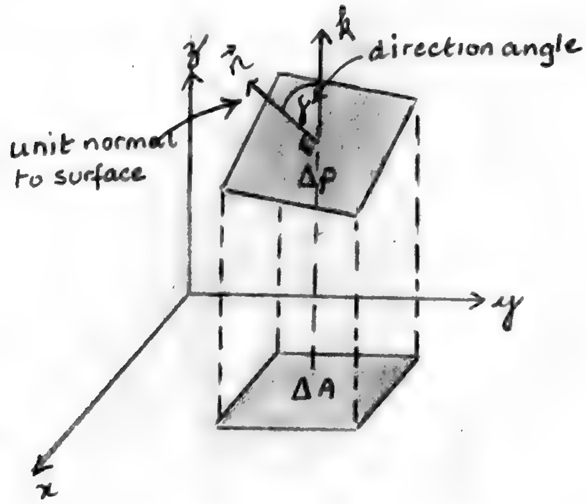
Now $P = z^2$; $Q = 2y$; $R = 2xz$ and $P_z = 2z$, $P_y = 0 = Q_x = R_y = Q_z$, $R_x = 2z$. Hence, $Q_x = P_y$ $R_x = P_z$ and $R_y = Q_z$ and the path of integration is immaterial. Choose the straight line path from (0,1,0) to (3,2,-1), then $x/3 = y - 1/1 = z/-1$ or $[0,1,0] + t[3,1,-1]$, and substituting in the integral using the 1st equations for the line, we have: $\int_0^3 x^2/9 + 2\int_1^2 ydy - 6\int_0^{-1} z^2dz = 6$. This result is easily verified by using the parametric form of the line and substituting for t; i.e., we have: $x = 3t$, $y = t + 1$, $z = -t$. $dx = 3dt$, $dy = dt$, $dz = -dt$. When $x = 0$, $t = 0$; when $y = 1$, $t = 0$; when $z = 0$, $t = 0$. Also, when $x = 3$, $t = 1$; when $y = 2$, $t = 1$; when $z = -1$, $t = 1$. Therefore, we have: $\int_0^1 (3t^2 + 2(2t + 1) + 6t^2)dt = t^3 + t^2 + 2t + 2t^3\Big]_0^1 = 6$.

EXERCISES(32):

1. Find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ by using line integrals.
2. Find by line integrals the area between $y^2 = 9x$ and $y = 3x$.
3. Find $\oint_C x^2ydx + ydy$ where C is the closed curve formed by $y^2 = x$ and $y = x$ between (0,0) and (1,1) - verify by using Green's theorem.
4. Find $\oint_C (x^2 + y)dx + (x - y^2)dy$ where C is the curve formed by $y^3 = x^2$ and $y = x$ between (0,0) and (1,1).
5. Show that $\int_{(0,1)}^{(1,2)} (x^2 + y^2)dx + 2xydy$ is independent of the path and find its value.
6. Evaluate $\oint_C (2x^3 - y^3)dx + (x^3 + y^3)dy$ around the circle $x^2 + y^2 = 1$.
7. Evaluate $\int_{(0,0)}^{(1,1)}$ xds over the line $y = x$.
8. Show that $\int_{(1,1,2)}^{(3,5,0)}$ yzdx + xzdy + xydz contains an exact differential and evaluate the line integral.
9. Find the area of the four-cusped hypocycloid $x = a\cos^3t$, $y = a\sin^3t$ by line integrals.
10. Evaluate $\int_C (xy + z^2)ds$ where C is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ which joins the points (1,0,0) and $(-1,0,\pi)$.

72. THE SURFACE INTEGRAL

Before extending the above concepts vectorially any more to 3-space, we must discuss the surface integral, since we will be dealing with this type of integral in both the divergence and Stokes' theorems. Let us then consider a surface in 3-space which we can project onto any of the 3 planes - i.e., the xy-plane, the xz-plane or the yz-plane. In the illustration below, we are projecting an infintesimal element of area ΔP onto the xy-plane. We call this infintesimal projected element of area ΔA and R indicates the total area onto which some surface area of a curve might be projected. R, of course, is a rectangular surface area since it is determined by one of the aforementioned planes.



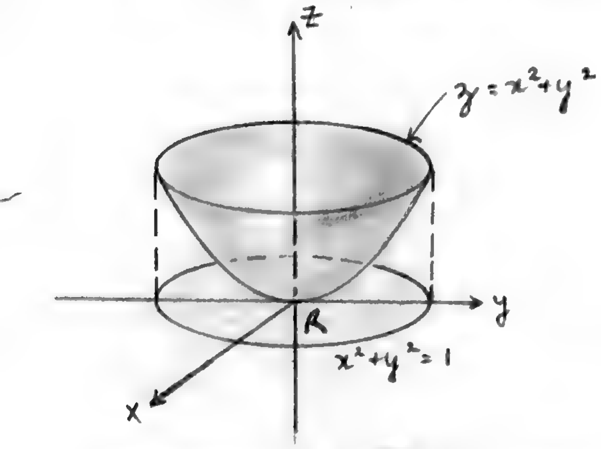
Now $dS \approx \Delta P$, but $\dfrac{\Delta P}{\Delta A} = \sec\gamma$.

From this we may obtain the expression:

$\Delta A = \cos\gamma \, \Delta P$ and thus, $dA = \cos\gamma dS$. But $dA = dxdy$ in this case, so that $dxdy = \cos\gamma dS$. Similarly, we can show by other projections that $dydz = \cos\alpha dS$ and $dxdz = \cos\beta dS$. Now $\cos\gamma = \dfrac{\vec{n}\cdot\vec{k}}{|n||k|}$ $= \vec{n}\cdot\vec{k}$ from the dot product formula. $\vec{n}$ can be found from the function $f(x,y,z)$, i.e., $\vec{n} = \nabla f/|\nabla f|$.

From the above, then, we have that $S = \iint_R dA/\cos\gamma$ (presuming that we project on the xy-plane).

<u>EXAMPLE 1</u>: Find the surface area of the paraboloid $z = x^2 + y^2$ below $z = 1$. (This projects onto $x^2 + y^2 = 1$ on the xy-plane as shown).



Now $S = \iint_R dA/\cos\gamma$ and $f(x,y,z) = x^2 + y^2 - z$.

Therefore $\nabla f = [2x, 2y, -1]$ and $\cos\gamma = \dfrac{-1}{\sqrt{4x^2 + 4y^2 + 1}}$

Hence, $S = \displaystyle\iint_R \frac{dxdy}{\frac{1}{\sqrt{4x^2 + 4y^2 + 1}}} =$

$\displaystyle\iint_{x^2+y^2\le 1} \sqrt{4x^2 + 4y^2 + 1}\, dxdy$. Change to polar coordinates so that $dA = rdrd\theta$ and $r^2 = x^2 + y^2$.

Therefore $S = \displaystyle\iint_{r\le 1} \sqrt{4r^2 + 1}\, rdrd\theta = \int_0^{2\pi}\int_0^1 \sqrt{4r^2 + 1}\, rdrd\theta = \pi/6(5\sqrt{5} - 1)$.

<u>EXERCISES(33)</u>:

1. Evaluate $\iint_S f(x,y,z)dS$ where S is the surface of the paraboloid $z = 2 - (x^2 + y^2)$ and $f(x,y,z) = 1$.

2. Evaluate $\iint_S (x^2 + y^2)dS$ where S is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $z = 3$.

3. Find the area of the surface of the sphere $x^2 + y^2 + z^2 = 4$ cut off by the cylinder $x^2 - 2x + y^2 = 0$.

4. Find the area of the surface of the sphere $x^2 + y^2 + z^2 = 25$ that lies in the 1st octant.

73. <u>JACOBIANS</u>

In the last example above, we switched from rectangular to polar coordinates to facilitate the integration. In so doing, we substituted the element $dA = rdrd\theta$ in polar coordinates for $dA = dxdy$ in rectangular coordinates. The question might

arise as to how one knows what the element of area is in polar coordinates and is there a general method for passing from one set of coordinates to another when one wants to deal with area or volume integrals? The answer to both these questions is in the affirmative and to arrive at the solution we must recall the concepts of the total differential, implicit functions and the theory of equations.

One can also use jacobians to find partial derivatives of composite functions that would otherwise prove rather unwieldy as will be shown. Let us suppose that we have two implicit functions $f(x,y,u,v) = 0$ and $g(x,y,u,v) = 0$ given, and further that we are given that $u = u(x,y)$ and $v = v(x,y)$, i.e. u and v are defined in terms of x and y. Now from elementary calculus, we know that the total derivatives must be:

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = f_x dx + f_y dy + f_u du + f_v dv$ and that

$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv = g_x dx + g_y dy + g_u du + g_v dv$ where we use the short notation $f_x = \frac{\partial f}{\partial x}$ and $g_x = \frac{\partial g}{\partial x}$ etc. Now since u and v are functions of x and y, in this case, we also have: $du = u_x dx + u_y dy$; $dv = v_x dx + v_y dy$ and we may substitute for du and dv in the first equations. We do this and we obtain:

$$df = f_x dx + f_y dy + f_u(u_x dx + u_y dy) + f_v(v_x dx + v_y dy).$$

$$dg = g_x dx + g_y dy + g_u(u_x dx + u_y dy) + g_v(v_x dx + v_y dy).$$

We then find that by rearranging terms, we will obtain the expression:

$$df = (f_x + f_u u_x + f_v v_x)dx + (f_y + f_u u_y + f_v v_y)dy.$$

$$dg = (g_x + g_u u_x + g_v v_x)dx + (g_y + g_u u_y + g_v v_y)dy.$$

But, since $f(x,y,u,v) = 0$ and $g(x,y,u,v) = 0$, certainly $df = 0 = dg$, therefore we have,

$f_x + f_u u_x + f_v v_x = 0$; $f_y + f_u v_y + f_v v_y = 0$ Thus, $f_u u_x + f_v v_x = -f_x$; $f_u u_y + f_v v_y = -f_y$

$g_x + g_u u_x + g_v v_x = 0$; $g_y + g_u u_y + g_v v_y = 0$ $g_u u_x + g_v v_x = -g_x$; $g_u u_y + g_v v_y = -g_y$

Solving for u_x, say, by Cramer's rule, we have the determinant of the system which is:

$\Delta = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}$ and therefore $u_x = -\dfrac{\begin{vmatrix} f_x & f_v \\ g_x & g_v \end{vmatrix}}{\Delta}$ assuming that $\Delta \neq 0$.

The determinant Δ is called the <u>jacobian</u> of f and g with respect to u and v. There are two well-known symbols of abbreviation for this determinant. They are $J\left(\frac{f,g}{u,v}\right)$ or $\frac{\partial(f,g)}{\partial(u,v)}$ The jacobian u_x, of course, is $J\left(\frac{f,g}{x,v}\right)$ or $\frac{\partial(f,g)}{\partial(x,v)}$ Extensions to other sets of implicit functions are possible and from the chapter on the theory of equations, we know that given m equations in n unknown where m is less than n, it is always possible

to solve for m of the variables in terms of the remaining n - m variables, i.e., the number of dependent variables equals the number of equations! We may make a summary of the more common types of problems using jacobians.

Case 1: 1 equation in 2 unknowns; $f(x,y) = 0$.

dependent variables

$$\frac{dy}{dx} = -f_x/f_y$$

independent variable

Case 2: 1 equation in 3 unknowns; $f(x,y,z) = 0$.

dependent — independent — dependent — independent

$$z_x = -f_x/f_z \qquad z_y = -f_y/f_z \qquad \text{where } f_z \neq 0.$$

dependent — dependent

Case 3: 2 equations in 3 unknowns; $f(x,y,z) = 0$; $g(x,y,z) = 0$.

dependent

$$\frac{dz}{dx} = -\frac{J\left(\frac{f,g}{y,x}\right)}{J\left(\frac{f,g}{y,z}\right)} = -\frac{\frac{\partial(f,g)}{\partial(y,x)}}{\frac{\partial(f,g)}{\partial(y,z)}} \qquad \frac{dy}{dx} = -\frac{J\left(\frac{f,g}{x,z}\right)}{J\left(\frac{f,g}{y,z}\right)}$$

independent — dependent variables — independent variables — dependent variables

Case 4: 2 equations in 4 unknowns; $f(x,y,u,v) = 0$; $g(x,y,u,v) = 0$.

dependent — dependent

$$\frac{\partial x}{\partial u} = -\frac{J\left(\frac{f,g}{u,y}\right)}{J\left(\frac{f,g}{x,y}\right)}; \quad \frac{\partial x}{\partial v} = -\frac{J\left(\frac{f,g}{v,y}\right)}{J\left(\frac{f,g}{x,y}\right)}; \quad \frac{\partial y}{\partial u} = -\frac{J\left(\frac{f,g}{x,u}\right)}{J\left(\frac{f,g}{x,y}\right)} \quad \text{etc.}$$

independent — dependent — independent

Case 5: 3 equations in 5 unknowns; $f(x,y,z,u,v) = 0$; $g(x,y,z,u,v) = 0$; $h(x,y,z,u,v) = 0$.

dependent

$$\frac{\partial x}{\partial u} = -\frac{J\left(\frac{f,g,h}{u,y,z}\right)}{J\left(\frac{f,g,h}{x,y,z}\right)} \quad \text{etc.}$$

independent — dependent variables

Now when the number of dependent variables equals the number of independent variables, the simultaneous equations can be regarded as a transformation of coordinates and it can be shown that in the case of 2-space the integral for area becomes:

$$\int_R f(x,y)\,dA = \iint_R f\left[x(u,v),\, y(u,v)\right]\left|J\left(\frac{x,y}{u,v}\right)\right| du\,dv$$

$$\int_R f(x,y,z)\,dV = \iiint_R f\left[x(u,v,w),\, y(u,v,w),\, z(u,v,w)\right]\left|J\left(\frac{x,y,z}{u,v,w}\right)\right| du\,dv\,dw.$$

These formulas look formidable but some examples will show just how useful jacobians are. Note that in the formulas that the dependent variables are always on top of the partial derivative and must occur in Δ.

EXAMPLE 1: Find the element of area $dA = dxdy$ in polar coordinates.

We recall that $x = r\cos\theta$; $y = r\sin\theta$ in polar coordinates. According to the above,

$dA = \left|J\left(\frac{x,y}{r,\theta}\right)\right|$ so that we immediately obtain $\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$. Thus $dA = rdrd\theta$.

EXAMPLE 2: Find the element of volume $dV = dxdydz$ in spherical coordinates.

Recall that spherical coordinates are given by $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$.

$$\text{Now } J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = \begin{aligned} &-r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi) \\ &-r\sin\theta\cos\phi(-r\sin^2\theta\cos\phi - r\cos^2\theta\cos\phi) \end{aligned}$$

$= r^2\sin\theta\sin^2\phi + r^2\sin\theta\cos^2\phi = r^2\sin\theta$. Therefore $dV = r^2\sin\theta drd\theta d\phi$. (Note that the third column of the determinant was used for evaluation).

EXAMPLE 3: Given that $f = 2x + y - 3z - 2u = 0$; $g = x + 2y + z + u = 0$. Find (1) $\left(\frac{\partial x}{\partial y}\right)_z$ (2) $\left(\frac{\partial z}{\partial u}\right)_x$ (3) $\left(\frac{\partial y}{\partial x}\right)_u$

$$\Delta = J\left(\frac{f,g}{x,u}\right) = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 4$$ since y and z are independent and hence, x,u must be dependent.

$$\text{Now, } \left(\frac{\partial x}{\partial y}\right)_z = \frac{-J\left(\frac{f,g}{y,u}\right)}{\Delta} = -\frac{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{\Delta} = -5/4$$

(2) Here u,x are independent, therefore, z,y are dependent. Accordingly,

$$\Delta = J\left(\frac{f,g}{z,y}\right) = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = 7$$

$$\left(\frac{\partial z}{\partial u}\right)_x = -\frac{J\left(\frac{f,g}{u,y}\right)}{\Delta} = -\frac{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}{\Delta} = -5/7$$

(3) In this case x,u are independent, therefore y,z are dependent and $\Delta = 7$. Accordingly,

$$\left(\frac{\partial y}{\partial x}\right)_u = -\frac{J\left(\frac{f,g}{z,x}\right)}{\Delta} = -\frac{\begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}}{\Delta} = -5/7$$

EXAMPLE 4: If $z = x^2 + y^3$ and $x = \sin(u - v)$, $y = uv^2$; find z_u and z_v.

Here it is clear that u,v are independent and x,y,z are dependent. Now $f = x^2 + y^2 - z = 0$; $g = x - \sin(u - v) = 0$; $h = y - uv^2 = 0$.

$$\Delta = J\left(\frac{f,g,h}{x,y,z}\right) = \begin{vmatrix} 2x & 3y^2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -1; \quad z_u = \frac{-J\left(\frac{f,g,h}{x,y,u}\right)}{\Delta} = \frac{\begin{vmatrix} 2x & 3y^2 & 0 \\ 1 & 0 & -\cos(u-v) \\ 0 & 1 & v^2 \end{vmatrix}}{1} =$$

$2x\cos(u - v) + 3y^2v^2 = 2\sin(u-v)\cos(u-v) + 3(uv^2)^2v^2 = \underline{\sin 2(u - v) + 3u^2v^6}$.

$$z_v = \frac{-J\left(\frac{f,g,h}{x,y,v}\right)}{\Delta} = -\frac{\begin{vmatrix} 2x & 3y^2 & 0 \\ 1 & 0 & \cos(u-v) \\ 0 & 1 & -2uv \end{vmatrix}}{-1} = -2x\cos(u - v) + 6y^2uv = \underline{-\sin 2(u-v) + 6u^3v^5}.$$

In the event that the jacobian of the transformation Δ is zero, we see immediately that we cannot proceed as in the above examples. What this means is, that there must exist some relationship between the functions involved. The proof of this is fairly straight-forward but will not be dealt with here. Instead, it might be instructive to look at an example to verify this functional relationship.

EXAMPLE 5: If $f(x,y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$ and $g(x,y) = \sin^{-1}x + \sin^{-1}y$, determine whether f and g are functionally related.

$$\text{Now } J\left(\frac{f,g}{x,y}\right) = \begin{vmatrix} \sqrt{1 - y^2} - \dfrac{xy}{\sqrt{1 - x^2}} & \sqrt{1 - x^2} - \dfrac{xy}{\sqrt{1 - y^2}} \\ \dfrac{1}{\sqrt{1 - x^2}} & \dfrac{1}{\sqrt{1 - y^2}} \end{vmatrix} = 0.$$

Thus, there is a functional relationship. Let $A = \sin^{-1}x$; $B = \sin^{-1}y$. Hence $x = \sin A$, $B = \sin y$. $\sin(A + B) = \sin A\cos B + \cos A\sin B = x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = f$, i.e., $f = \sin g$.

It might be noted that if $u = f(x,y)$; $v = g(x,y)$ then the vector equivalent form which would show functional dependence would be that $\nabla f \times \nabla g = 0$ (i.e., $\nabla u \times \nabla v = 0$). Also, if $u = f(x,y,z)$; $v = g(x,y,z)$; $w = h(x,y,z)$, then $\nabla f \cdot \nabla g \times \nabla h = 0$ iff u,v,w are functionally dependent.

EXERCISES(34):

1. If $xu^2 + v = y^3$ and $2yu - xv^3 = 4x$, find (a) u_x (b) v_y.
2. If $x = r\cos\theta$, $y = r\sin\theta$, find r_x and θ_x.
3. If $z = u^2 + v^2$, $u = r\cos\theta$, $v = r\sin\theta$; find z_r and z_θ.
4. Given $u = \ln\sqrt{x^2 + y^2}$, $v = \tan^{-1}y/x$, prove that $\left(\frac{du}{dx}\right)_y\left(\frac{dx}{du}\right)_v + \left(\frac{du}{dy}\right)_x\left(\frac{dy}{du}\right)_v = 1$.
5. Given that $u^3 + v^3 + x^3 - 3y = 0$; $u^2 + v^2 + y^2 + 2x = 0$, find $\left(\frac{du}{dx}\right)_y$
6. Given that $x^2 - y\cos(uv) + z^2 = 0$; $x^2 + y^2 - \sin(uv) + 2z^2 = 2$; $xy - \sin u\cos v + z = 0$, find, $\left(\frac{\partial z}{\partial u}\right)_v$ & $\left(\frac{\partial x}{\partial v}\right)_u$ at the point $x = 1$, $y = 1$, $u = \pi/2$, $v = 0$, $z = 0$.
7. If $x = u^2 - v^2$, $y^2 = 2uv$, compute (a) the jacobian of the transformation (b) $\left(\frac{\partial u}{\partial x}\right)_y$
8. Find the element of volume in cylindrical coordinates where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
9. Determine whether $f(x,y) = e^x\sin y$ and $g(x,y) = x + \ln\sin y$ are functionally dependent. If so, indicate the relationship.
10. Prove that if $x = f(u,v)$, $y = g(u,v)$, then $\left(\frac{\partial x}{\partial y}\right)_u\left(\frac{\partial y}{\partial u}\right)_u = 1$.

CHAPTER 14 - DIVERGENCE AND STOKES' THEOREM

74. DIVERGENCE THEOREM

Essentially the divergence theorem is an extension of Green's theorem from 2-space to 3-space. It is also known as Gauss' theorem or Green's theorem in space. The proof

will not be given here since almost any text on vector calculus will contain an adequate proof. Basically, the divergence theorem relates volume and surface integrals and if one thinks of a surface which encloses some sources and sinks, then the total normal flow(sometimes called flux) of the physical entity over the surface can be measured in two ways: (1) by summing the outward normal flux over the surface (2) by computing the algebraic sum of the sources and sinks throughout the volume. This can best be described by stating that in a vector field the surface integral of the normal component of the flux(the perpendicular component to the surface) is equal to the volume integral of the divergence taken throughout the volume. Needless to say there are certain mathematical restrictions that have to be observed. These are that the first partial derivatives of the components $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ of the vector A have to be single-valued and continuous in the region T bounded by the particular closed surface S. Intuitively the surface must not be cut in more than two places by any straight line parallel to the coordinate axes. If this is not the case, we must "doctor" the surface until it meets the aforementioned conditions. Mathematically then, the relation between surface and volume integrals takes on the short vector form:

$\int_V(\nabla\cdot A)dV = \int_S A\cdot d\vec{s}$ ($d\vec{s}$ is the unit normal to the surface in question).

The meaning might become clearer if we write the above in the cartesian forms: $A = [P,Q,R]$.

$$\iiint_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz = \iint_S Pdydz + Qdxdz + Rdxdy = \iint_S(P\cos\alpha + Q\cos\beta + R\cos\gamma)dS,$$

where we recall that $dS\cos\gamma = dxdy$; $dS\cos\beta = dxdz$; $dS\cos\alpha = dydz$ or in vector form:

$d\vec{s} = [\cos\alpha ds, \cos\beta ds, \cos\gamma ds] = |d\vec{s}|[\cos\alpha, \cos\beta, \cos\gamma]$.

<u>EXAMPLE 1</u>: Evaluate $\iint_S A\cdot ds$ where $A = [x^3, x^2y, x^2z]$ is the surface of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$.

We immediately obtain by use of the divergence theorem that :

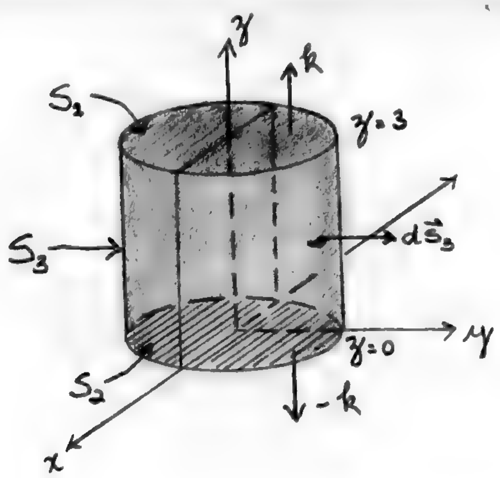
$\int_S A\cdot ds = \int_V(\nabla\cdot A)dV$. Thus $\nabla\cdot A = \nabla\cdot[x^3,x^2y,x^2z] = 3x^2 + x^2 + x^2 = 5x^2$.

$\int_V(\nabla\cdot A)dV = 4\cdot5\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 x^2dzdydx$ (taking the portion in the 1st quadrant and multiplying by 4).

This gives us: $60\int_0^2\int_0^{\sqrt{4-x^2}}x^2dydx = 60\int_0^2 x^2\sqrt{4-x^2}dx = \Big|_0^2 60(-x/4(4x^2)^{3/2} + \frac{1}{2}(x\sqrt{4-x^2} + 4\sin x/a))$

$= (\frac{1}{2}(4\pi/2)) = 60\pi$.

Using surface integrals, we see from the figure below that we have three surface integrals to calculate - indicated by the respective surfaces S_1, S_2, S_3.



$S_1 = \iint_S A\cdot k ds = \iint_S [x^3, x^2y, zx^2]\cdot[0,0,1]ds$

$= \iint x^2 z dxdy$ (since $\cos\gamma = 1$; $\gamma = 0$!). Now $z = 3$

on S_1 so that we have: $S_1 = 3\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 dxdy$

$= 2\int_{-2}^{2}(\sqrt{4-y^2})^3 dy = 2\ 2(3\cdot 16/8\cdot \pi/2) = 12\pi.$

$S_2 = 0$ since $z = 0$.

$S_3 = 2\iint_S A\cdot d\vec{S}_3$ (where we must project <u>both</u> halves of the surface!).

$d\vec{S}_3 = \vec{n} dS_3$ where $n = \nabla f/|\nabla f| = \dfrac{[2x,2y,0]}{2\sqrt{x^2+y^2}} = \dfrac{[x,y,0]}{\sqrt{x^2+y^2}}$

We may project surface area S_3 either on the xz-plane or the yz-plane but it is not possible to project it on the xy-plane. We choose the yz plane. Recall that $dS\cos\alpha = dydz$. Hence, $dS = dydz/\cos\alpha$. Now $\cos\alpha = x/\sqrt{x^2+y^2}$ from the above. Therefore we have:

$S_3 = 2\int_0^3\int_{-2}^{2}\left([x^3,x^2y,x^2z]\cdot\dfrac{[x,y,0]}{\sqrt{x^2+y^2}}\right)\left(\dfrac{\sqrt{x^2+y^2}}{x}\right)dydz = 2\int_0^3\int_{-2}^{2}(x^3+xy^2)dydz$

$= 2\int_0^3\int_{-2}^{2}(\sqrt{4-y^2})^3 dydz = 2\int_0^3\int_{-2}^{2} y^2\sqrt{4-y^2}dydz = 2\int_0^3 2(3\cdot 2^4/8\cdot\pi/2)dz + 2\int_0^3(2^4/8\cdot\pi/2)dz$

$= 36\pi + 12\pi$. Therefore $S_1 + S_2 + S_3 = 12\pi + 0 + 36\pi + 12\pi = 60\pi$ and this verifies the first result!

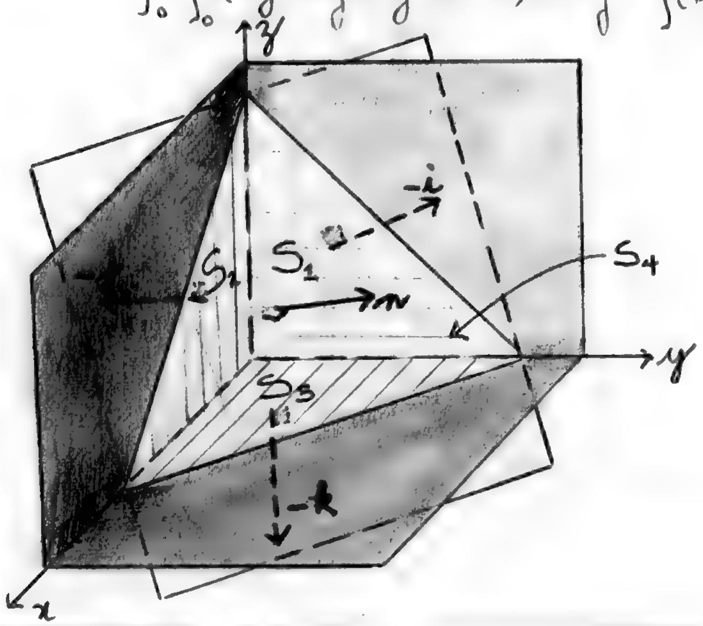
<u>EXAMPLE 2</u>: Verify the divergence theorem where we wish to evaluate the surface integral $\iint_S xydydz - x^2dxdz + (x+z)dxdy$ over the surface S which is that portion of the plane $2x + 2y + z = 6$ included in the 1st octant. Find the common value.

Here $A = [xy, -x^2, x+z]$ and using the divergence theorem, we have:

$\iint_S A\cdot ds = \iiint(\nabla\cdot A)dv = \int_0^3\int_0^{3-y}\int_0^{6-2x-2y}(y+1)dzdxdy = \int_0^3\int_0^{3-y}(y+1)(6-2x-2y)dxdy$

$= \int_0^3\int_0^{3-y}(4y-2xy-2y^2+6-2x)dxdy = \int(4xy - x^2y - 2y^2x + 6x - x^2)_0^{3-y}dy = \int_0^3 4(3-y)y-(3-y)^2y-2y^2(3-y)-6(3-y)-(3-y)^2dy$

$= \int_0^3(9+3y-5y^2+y^3)dy = \left[9y + \frac{3y^2}{2} - \frac{5y^3}{3} + \frac{y^4}{4}\right]_0^3 = 63/4.$



We see from the figure that we have 4 surface integrals to calculate - S_1, S_2, S_3, S_4.

Now $S_1 = \iint_S [xy,-x^2,x+z]\cdot[-1,0,0]dzdy$ (since $\alpha = 0$). Thus, $S_1 = \iint_S xydzdy = 0$ (since $x = 0$).

$S_2 = \iint_S [xy,-x^2,x+z]\cdot[0,-1,0]dxdz$ (since $\beta = 0$).

Thus $S_2 = \iint_S x^2dxdz = \int_0^6\int_0^{\frac{6-z}{2}} x^2dxdz = 27/2.$

$S_3 = \iint_S [xy,-x^2,x+z]\cdot[0,0,-1]dxdy$ (since $\gamma = 0$).

$$S_3 = \iint_S -(x+z)\,dx\,dy = -\int_0^3\int_0^{3-y} x\,dx\,dy = -9/2$$

$S_4 = \iint_S [xy, -x^2, x+z]\cdot\vec{n}\,ds$ and we will project the surface on the xy-plane.

$\vec{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{[2,2,1]}{3}$. Now $dS\cos\gamma = dxdy$ but $\cos\gamma = 1/3$

Therefore $S_4 = \iint_S [xy,-x^2,x+z]\cdot[2,2,1]/3\ dxdy/1/3 = \iint_S (2xy - 2x^2 + x + 6 - 2x - 2y)dxdy$

since $z = 6 - 2x - 2y$. Therefore we have:

$$S_4 = \int_0^3\int_0^{3-x}(2xy - 2x^2 + x - 6 - 2x - 2y)dxdy = 27/4.$$

Now $S_1 + S_2 + S_3 + S_4 = 0 + 27/2 - 9/2 + 27/4 = 63/4$ which verifies the 1st result!

As one can readily see the integration is arduous and quite tedious but the theorem proves useful in a variety of engineering applications where one wishes to flit from a surface to a volume integral or vice versa.

75. STOKES' THEOREM

Just as we related surface and volume integrals in the case of the divergence theorem, we will now relate surface and line integrals by means of Stokes' theorem. Essentially this theorem states that the line integral of the tangential component of any vector A around a closed path is equal to the surface integral of the normal component of the curl of A over the surface enclosed by the path. Symbolically, we write:

$\int_C \vec{A}\cdot d\vec{r} = \int_S (\nabla\times\vec{A})\cdot d\vec{s}$ ($d\vec{s}$ is the unit normal to the surface).

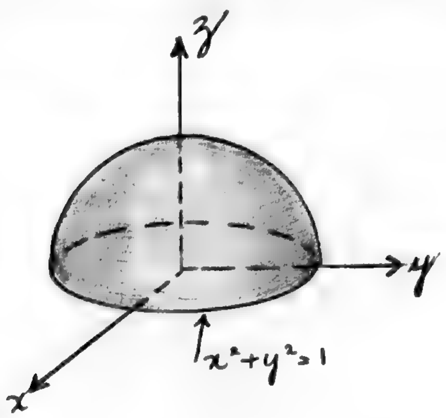
Again, we will assume the theorem without proof and the interested student will find a plethora of texts to consult for said proof. Other forms of the theorem are:

$$\begin{aligned}\int_C P dx + Q dy + R dz &= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dxdz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \\ &= \iint_S\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right]dS\end{aligned}$$ where $\cos\gamma dS = dxdy$ etc.

Recall that in Chapter 13, section 71, we assumed that a line integral was independent of the path iff $\nabla\times A = 0$ and we deferred the proof until the discussion of Stokes' theorem. We can now quickly see from the short form of the theorem, viz., $\int_C A\cdot dr = \int_S (\nabla\times A)\cdot ds$ that if $\int_C A\cdot dr$ is independent of the path then $\oint_C A\cdot dr = 0$ (around any closed path) which immediately indicates that $\int_S (\nabla\times A)\cdot ds = 0$, but this means that $\nabla\times A$ must equal zero.

<u>EXAMPLE 1</u>: Verify Stokes' theorem for $A = [2x - y, -yz^2, -y^2z]$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Now the boundary of C is a circle in the xy-plane of radius 1 and whose center is at



the origin. Therefore the equation will be $x^2 + y^2 = 1$. To evaluate the line integral $\int_C A \cdot dr$, we must change to a parameter t. Thus, we have x = cost, y = sint, z = 0 and the position vector r = [cost,sint,0] which yields: dr = [-sint,cost,0]dt. Therefore, we have:

$\int_C A \cdot dr = \int_0^{2\pi} [2\cos t - \sin t,0,0] \cdot [-\sin t,\cos t,0]dt$
$= \int_0^{2\pi}(-2\sin t\cos t + \sin^2 t)dt.$

Now to facilitate integration, we must recall formulas of definite integration of trigonometric functions where the limits of integration are taken over an interval of 2π. These formulas are listed below for convenience:

(1) $\int_C^{C + 2\pi} \sin(nx)dx = \int_C^{C + 2\pi} \cos(nx)dx = 0,\ n \neq 0.$

(2) $\int_C^{C + 2\pi} \sin(mx)\cos(nx)dx = 0.$

(3) $\int_C^{C + 2\pi} \sin(mx)\sin(nx)dx = \int_C^{C + 2\pi} \cos(mx)\cos(nx)dx = 0,\ m \neq n.$

(4) $\int_C^{C + 2\pi} \sin^2(nx)dx = \int_C^{C + 2\pi} \cos^2(nx) = \pi,\ n \neq 0.$

Using formulas (2) and (4) to complete the example, we see that $\int_C A \cdot dr = \int_0^{2\pi} -2\sin t\cos t dt + \int_0^{2\pi} \sin^2 t dt = \pi.$

Now $\nabla \times A = \begin{vmatrix} i & \frac{\partial}{\partial x} & 2x - y \\ j & \frac{\partial}{\partial y} & -yz^2 \\ k & \frac{\partial}{\partial z} & -y^2 z \end{vmatrix} = [0,0,1]$. If we project the surface onto the xy-plane, we will use $\cos\gamma$.

$n = \dfrac{\nabla f}{|\nabla f|} = \dfrac{[x,y,z]}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{[x,y,z]}{1}$; Thus $\cos\gamma = z/1$ and $dxdy/z = dS$.

$\iint_S (\nabla \times A) \cdot ds = \iint_S [0,0,1] \cdot \dfrac{[x,y,z]}{1} \dfrac{dxdy}{z} = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = 2\int_{-1}^{1}\sqrt{1 - y^2}dy = \pi,$

which verifies the 1st result.

<u>EXAMPLE 2</u>: Evaluate $\iint_S (\nabla \times A) \cdot ds$ where $A = [x - z, x^3 + yz, -3xy^2]$ and S is the surface above the xy-plane of the cone $z = 2 - \sqrt{x^2 + y^2}$.

We may evaluate directly or we may use Stokes' theorem. In the latter case, we have: $\iint_S (\nabla \times A) \cdot ds = \int_C A \cdot dr$. We change to parameter t and we know that the boundary of C in the xy-plane is a circle $x^2 + y^2 = 4$(when z = 0). Thus, we let x = 2cost, y = 2sint, z = 0. Therefore r = [2cost,2sint,0] and dr = [-2sint,2cost,0]dt. Substituting for x,y,z in A, we have $A = [2\cos t, 8\cos^3 t, -3\cos t\sin^2 t]$. We now substitute in the left hand side of

Stokes' theorem, we have :

$$\int_0^{2\pi} A\cdot dr = \int_0^{2\pi} [2\cos t, 8\cos^3 t, -3\cos t\sin^2 t]\cdot[-2\sin t, 2\cos t, 0]dt = \int_0^{2\pi}(4\cos t\sin t - 16\cos^4 t)dt$$

$$= -\int_0^{2\pi} 4\cos t\sin t\,dt + 16\int_0^{2\pi}\cos^4 t\,dt = 0 + 16[3t/8 + \sin 2t/4 + \sin 4t/32]_0^{2\pi} = 16[3t/8]_0^{2\pi} = 12\pi$$

A similar result may be obtained by use of the surface integral but with considerably more effort!

EXERCISES(35):

1. Verify the divergence theorem for $A = [2xy + z, y^2, -(x + 3y)]$ taken over the region bounded by $2x + 2y + z = 6$; $x = 0$, $y = 0$, $z = 0$.

2. Determine the value of $\int\int_S xdydz + ydzdx + zdxdy$ where S is the surface of the region bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3$ by use of the divergence theorem.

3. Verify Stokes' theorem for $A = [2y, 3x, -z^2]$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is its boundary.

4. Compute $\int_S A\cdot ds$ where S is the surface of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $z = 1$ where $A = [x, -y, z]$.

5. Evaluate $\int_S r\cdot ds$ where r is the position vector of the points on the surface of the ellipsoid $x^2/1 + y^2/4 + z^2/9 = 1$.

6. If $A = [y, z, -x]$ and S is the surface of the paraboloid $z = 1 - x^2 - y^2$ where z is greater or equal to zero, evaluate $\int_S(\nabla \times A)\cdot d\vec{s}$.

7. Evaluate $\int_S (\nabla \times A)\cdot d\vec{s}$ if $A = [y^2, xy, -xz]$ and S is the hemisphere $x^2 + y^2 + z^2 = 1$ where z is greater than or equal to zero.

8. Evaluate $\int_C A\cdot dr$ if $A = [x^2 + y^2, x^2 + z^2, y]$ and C is the circle $x^2 + y^2 = 4$ in the plane $z = 0$.

9. By applying the divergence theorem to ϕA where A is an arbitrary constant vector, show that $\int_S \phi d\vec{s} = \int_V \nabla\phi dV$ (hint: the dot product of a constant vector may be removed from the integral.

10. By applying Stokes' theorem to the vector ϕA where A is any arbitrary constant vector, show that $\int_C \phi d\vec{r} = -\int_S \nabla\phi \times d\vec{s}$.

ANSWERS TO EXERCISES

EXERCISE 1 (page 6) (1) hyperbola (2) ellipse (3) ellipse (4) hyperbola (5) parabola (6) hyperbola (7) ellipse (8) 2 straight lines (9) parabola (10) 2 straight lines (11) 2 straight lines (12) 2 straight lines (13) ellipse (14) no points (15) hyperbola (16) 2 straight lines (17) 2 straight lines (18) 2 straight lines (19) parabola (20) hyperbola

EXERCISE 2 (page 12) (1) circle-center origin, $r = \sqrt{3}$ (2) parabola opening up, l.r. = 4. (3) hyperbola opening right and left, $c = \sqrt{13}$ (4) ellipse $a = 2$, $b = \sqrt{8}/\sqrt{3}$ (5) circle center $(-1/4, 1/2)$, $r = \sqrt{29}/4$ (6) point (0,0) (7) two straight lines $x = \pm y$ (8) imaginary ellipse(no graph) (9) hyperbola $a = 2$, $b = 3$, center $(2,-3)$ (10) parabola opening right, vertex at (0,2) (11) ellipse major axis 4, minor axis 2, center at (-1,2) (12) parabola opening up, vertex at (1,0), l.r. = 1 (13) null set (14) 2 straight lines$(x-1)/2 = \pm(y - 2)/3$ (15) $\sqrt{117}/2$ by $\sqrt{117}/2\sqrt{3}$ ellipse, center at $(5/2,-1)$ (16) parabola opening down, vertex at (1,2) (17) (2,-1) (18) $\sqrt{3}$ by 4 ellipse, center at (2,-1) (19) circle, center at 2,-2/3), radius = $\sqrt{58}/3$ (20) 2 by 3 hyperbola opening up and down, vertex at (-1,2).

EXERCISE 3 (page 15) 1. $3x^2 - 2y^2 - 6 = 0$. 2. $x' = \pm 1$. 3. $3x'^2 - 11y'^2 - 66 = 0$. 4. $4x'^2 - 9y'^2 = 36$. 5. $6x'^2 + y'^2 = 10$.

EXERCISE 4. (page 17) 1. $4x'^2 - y'^2 = 4$. 2. $x'^2 = 8y'$. 3. $x'^2 - 3x'y' + 3' = 0$. 4. $x'^2 - 3x'y' + y'^2 - 9 = 0$. 5. $y'^2 = 8x'$.

EXERCISE 5 (page 19) 1. $x'^2 - y'^2 = 8$. 2. $3x''^2 - y''^2 = 3$. 3. $2x''^2 + y''^2 = 2$. 4. $5x''^2 - 2y''^2 + 36 = 0$. 5. $x''^2 = 6y''$. 6. $3x''^2 - 2y''^2 + 6 = 0$. 7. $x''^2 + 4y'' = 0$. 8. $3x''^2 - 9y''^2 = 27$. 9. $4x''^2 + 2y''^2 - 1 = 0$. 10. $3x + y = 5$, $3y - x = 2$.

EXERCISE 6 (page 28) 1. z-axis. 2. ellipsoid 3. hyperbolic paraboloid 4. elliptic paraboloid 5. hyperboloid of 2 sheets 6. cone 7. hyperboloid of 2 sheets 8. 2 intersecting planes 9. elliptic cylinder 10. parabolic cylinder.

EXERCISE 7 (page 30) 1. hyperboloid of 2 sheets(-2,0,1). 2. ellipsoid(2,-2,2). 3. hyperbolic cylinder, $x = -3$, $z = 0$. 4. cone(1,0,-2). 5. elliptic paraboloid(2,-2,-1). 6. hyperbolic paraboloid-down-(2,2,-9). 7. Hyperbolic paraboloid(-3/2,3,-3). 8. 2 planes. $2x - 3y + z - 1 = 0 = 2x + y + 2z - 2$. 9. (1,2,0). 10. (367/231,45/77,173/231).11. $x^2 + y^2 + z^2 - 2x + 4y + 6z - 11 = 0$. 12. $4x^2 - 4y^2 + z^2 + 8x + 24y - 36 = 0$. 13. $4x^2 - 4y^2 - z^2 + 8x + 24y - 36 = 0$. 14. $x^2 + 16y^2 + z^2 - 4x + 32y = 5$. 15. elliptic paraboloid,(2,-2,-1), z-axis. 16. elliptic paraboloid(0,0,-2), y-axis. 19. cone(0,0,4), z-axis. 20. cone(-3,1,0), x-axis. 17. Hyperboloid of 1 sheet (-2,1,-1), z-axis. 18. Hyp.of 1 sheet (-1,2,3), y-axis.

EXERCISE 8 (page 31) 1. scalar 2. scalar 3. scalar 4. scalar 5. vector 6. scalar 7. vector 8. scalar 9. scalar 10. scalar 11. scalar 12. vector 13. vector 14. vector 15. vector 16. scalar 17. scalar 18. vector 19. vector 20. scalar.

EXERCISE 9 (page 34) 1. [1,8] 2. [6,-2,3] 3. [0,0,8] 4. BC 5. CB 6. 13 7. $\sqrt{6}$ 8. 9 9. 23 10. $\sqrt{69}$ 11. [-4,2] 12.[4,-2] 13. [1,-7] 14. [4,-6] 15. [-1,3] 16.[-3,1,-4] 17. [3,-1,4] 18. [-1,6,3] 19.[2,5,7] 20. [-1,6,3].

EXERCISE 10 (page 36) 1. [5,14] 2. $[-3\sqrt{5}, -4\sqrt{5}]$ 3. 5[-2,1] 4. $[-2/\sqrt{5} - 3/5,\ 1/\sqrt{5} - 4/5]$ 5. [8,18] 6. [-8,8,-19] 7. 30[1,-1,1] 8. [-3/2,3/2,-13/4] 9. 1 10. 3.

EXERCISE 11 (page 37) 1. t[-7/25,24/25] 2. t[5/31,6/31,-30/31] 3. [2,-3,0] 4. [2,-3] 5. 3i - 5j - k.

EXERCISE 12 (page 38) 1. 132.287, 79.11° 2. [0,4,-1], $\sqrt{17}$ 3. $\sqrt{3}$, 30° 4. 120 pounds,-32° 5. 1.8052 hours, 18 miles.

EXERCISE 13 (page 42) 1. (-11k,7k,5k) 2. (2,3,-1) 3. (-1,-1) 4. (-1,2,1) 5.coplanar(-3k,2k,k

EXERCISE 14 (page 43) 2. use L.D. 3. use Pythagorean thm. 4. use l.I. 7. use area.

EXERCISE 15 (pages 47-48) 1. 27 2. -33 3. 107 4. 0 5. 1 6. 0 7. abc(a-b)(b-c)(c-a) 8. -266 9. 0 10. -1562

EXERCISE 16 (page 50) 1. 2,2,2 2. not possible 3. $\begin{bmatrix} -4 & 10 & 0 \\ -13 & 4 & -15 \\ 8 & 21 & -24 \end{bmatrix}$ 4. -5196 5. $\begin{bmatrix} 10 & 28 \\ -27 & -18 \end{bmatrix}$

EXERCISE 17 (pages 57-58) 1. (a) 14 (b) -10 (c) -178 (d) $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 4 & 1 \\ -7 & 5 & 15 \end{bmatrix}$ (e) $\begin{bmatrix} 59 & -21 & -105 \\ -21 & 45 & 73 \\ -105 & 73 & 235 \end{bmatrix}$

(f) $\begin{bmatrix} 55 & -52 & 43 \\ -45 & -6 & -19 \\ 14 & -10 & -2 \end{bmatrix}$ (g) symmetric (h) $\begin{bmatrix} 14 & 8 & -42 \\ 8 & 26 & 14 \\ -42 & 14 & 299 \end{bmatrix}$ (i) $\begin{bmatrix} 2 & 5 & -10 \\ 5 & 8 & 6 \\ -10 & 6 & 30 \end{bmatrix}$ (j) -1742

2. $\begin{bmatrix} 2/34 & -4/34 \\ 10/34 & -3/34 \end{bmatrix}$ 3. none 4. $-2R_1 + R_2$, $3R_1 + R_3$, $R_3/5$, $-5C_1 + C_2$, $2C_1 + C_3$, $-C_1 + C_4$, $5R_3 + R_2$.

5. $\begin{bmatrix} -2 & 3 \\ -4 & 1 \end{bmatrix}$ 6. (a) 3 $\begin{bmatrix} -1/3 & 2/3 & -1/3 \\ 2/3 & -4/3 & 2/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$ (b) 0 (c) 1 (d) doesn't exist (e) symmetric

7. (a) 2 $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ (b) $\frac{1}{2}$ $\begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & -3 & 2 \\ -1 & 1 & -2 \\ 1 & 1 & -4 \end{bmatrix}$ (d) doesn't exist (e) $\begin{bmatrix} 6 & 4 \\ 4 & 10 \end{bmatrix}$

8. (a) A (b) 1 (c) A (d) I (e) $\begin{bmatrix} 34 & -7 & -37 \\ -7 & 2 & 7 \\ -37 & 7 & 41 \end{bmatrix}$ 9. (a) -3A' (b) A/9 10. $\dfrac{\begin{bmatrix} 3 & 6 & 3 & 3 \\ 0 & -10 & -6 & -2 \\ -3 & -14 & -9 & -1 \\ -3 & 2 & 3 & 1 \end{bmatrix}}{6}$

EXERCISE 18 (pages 66-67) 1. consistent 2. inconsistent 3. consistent 4. consistent 5. consistent 6. inconsistent 7. consistent 8. consistent 9. inconsistent 10. consistent 11. consistent 12. consistent 13. inconsistent 14. consistent 15. inconsistent 16. consistent 17. ~~in~~consistent 18 inconsistent 19. consistent 20. inconsistent.

EXERCISE 19 (page 69) 1. (1,-2) 2. no solution 3. (1,-1,0) 4. (k-1,1-k,-k) 5. (-1,k+3,k) 6. no solution 7. (2,-2,1) 8.(5k-7,5-3k,k) 9. no solution 10. (9k+13/3,-k,2k-1,-4/3) 11. (k,-2k,2k) 12. (2k,k) 13. (0,0) 14. (13k/7,-k/7,k) 15. (0,0,0) 16. (-k-5t/3,5k+t/3,k,-t) 17. (4k-16t,11t-3k,k,t) 18. (0,0,0,0) 19. (-26-84k/71,30+15k/71,133k+53/71,k) 20.(0,0,0,0).

EXERCISE 20 (pages 75-76) 1. (a) 21 (b) -2/3 (c) 4/21 (d) [-1/3,2/3,-2/3] (e) -26/63 2. (a) $\sqrt{35}/2$ (b) [$-1/\sqrt{35}$,$3/\sqrt{35}$,$5/\sqrt{35}$] (c) [-3,1,-1] (d) t[11,-3,4] (e) $\sqrt{11}$ 3. (a) [-119/121, 102/121, 102/121] (b) 17/11 4. (a) 3 (b) $\sqrt{14}$ (c) [1,-1,1] (d) 70.89° (e) $5\sqrt{3}$ 6.(a) [-21/35,28/35] (b) [-20/29,-21/29] (c) [1/3,2/3,+2/3] (d) [2/11,6/11,9/11] (e) [1/3,2/3,2/3] 5. (a) 5 (b) 17 (c) 15 (d) 17 (e) 9 7. $5\sqrt{3}$ 12. 19 13. $3\sqrt{2}$ 14. 11.44 15.(a) 2[9,-13,10] (b) 36[-3,1,3]

EXERCISE 21 (page 80) 2. (-3,2,1) 3. (a) 344 (b) 48.76 (c) 49/9, 49/81[-7,4,-4] (d) $[-252/\sqrt{120266}, 199/\sqrt{120266}, 131/\sqrt{120266}]$ (e) 2[168,-67,-361].

EXERCISE 22 (page 84) 1. (a) x+3/1 = y-1/-4, [-3,1] + t[1,-4] (b) x-5/0 = y+1/-1, [5,-1] + t[0,-1] (c) x+1/7 = y-3/-5 = z-4/-1, [-1,3,4] + t[7,-5,-1] (d) x/4 = y/-3 = z/2, t[4,-3,2] (e) x-2/2 = y+3/-1 = z-1/-2 2. (a) 1, -4; $1/\sqrt{17}$, $-4/\sqrt{17}$ (b) 0, -1; 0, -1 (c) 7,-5,-1 $7/\sqrt{75}, -5/\sqrt{75}, -1/\sqrt{75}$ (d) 4,-3,2; $4/\sqrt{29}, -3/\sqrt{29}, 2/\sqrt{29}$ (e) 2,-1,-2; 2/3,-1/3,-2/3 3. 14.03° 4. (a) 28.46° (b) 36.07° 5. [11,-2,3] + t[4,-9,1] 6. [-1,5,2] + t[-1,5,17] 7. (1,-3,2) 9. t[1,1,-1] 10. [-4,2,9] + t[-14,-33,10].

EXERCISE 23 (pages 89-90) 1. (a) 8x + 13y - 2z - 4 = 0 (b) 17x + 13h + 7z = 0 (c) x - 4 = 0 (d) 3x + y + 5z - 16 = 0 (e) 2x + z - 10 = 0 2. (a) 3x - 2y + 5z - 19 = 0 (b) 3x - 2z = -14 3. (a) 3x - 2y + 5z = 14 (b) z = 4 (c) 3x + 9y + z + 23 = 0 (d) infinite number of planes with [A,B,C] perpendicular to [2,3,-1] e.g. x - y - z - 4 = 0 = 2x - y + z - 9 etc. (e) 2x+3y -z+1=0 4. (a) x + 2y + z + 5 = 0 (b) 9x - 11y + 6z + 1 = 0 5. (a) 90° (b) 0° (c) 60° (d) 46.5° 6. (a) (-19/41,112/41,124/41) (b) (21/8,17/16,7/4) (c) x+1/3 = y+7/3/7/3 = z (d) no solution (e) (34/3,32/3,-16/3) 7. (a) 11x - 3y + 4z - 17 = 0 (b) 23x + 30y + 44z - 53 = 0 8. 5x - 13y + 4z - 6 = 0 = 23x - y - 32z - 78 9. (a) (-1,-6,3) (b) it doesn't (c) everywhere (lies in plane) (d) (4,7,-2) 11. (a) [2,-2,0] + t[5,40,22] (b) [2,5,0] + t[1,6,7] (c) t[1,1,-1] (d) [0,1,1] + t[1,-1,7] 11. (a) [1,2,3] + t[9,-21,-29] (b) [-1,0,2] + t[15,5,-7] 12. (a) [1,-2,-2] + t[1,-3,2] (b) [-1,0,0] + t[2,-1,8] 13. (a) [-2,3,-1] + t[2,-9,-12] (b) [1,2,3] + t[4,-3,-1], (1,2,3) (c) none such-lines don't meet (d) [1,2,3] + t[0,1,-3] 14. (a) 40.7° (b) 23.89° 15. (a) [5,-2,3] + t[1,1,1] (b) (3,-2,6), (33/7,-32/7,48/7

EXERCISE 24 (page 95-96) 1. (a) 8.55 (b) .6 (c) $17/\sqrt{13} = 4.71$ (d) 4.62 (e) 3.8 2. (a) 0 (b) 6.13 (c) 3.2 (d) 6.68 (e) 4.66 3. (a) 1.92 (b) .566 (c) 3 (d) .763 (e) 3.683 4. (a) (-37/27,44/27,-31/27) (b) (106/78,299/78,277/78) (c) (2,0,0) (d) (4/11,-6/55,7/55) (e) (378/62,661/62,17/62) 5. (a) $(x + 4)^2 + (y + 1)^2 + (z + 2)^2 = 36$ (b) $(x - 2)^2 + (y - 5)^2 + (z + 8)^2 = 122$ 6. (a) $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 121$ (b) 2x - 6y - 9z = 84 7. 23.48 8. (a) 1/3 (b) 3 9. (a) 3x - 3y + 10z + 5 = 0 (b) 5x + y + 6z - 19 = 0 10. no, skew (b) yes, non-skew.

EXERCISE 25 (pages 101-102) 1. 83A/70 + 2B/5 - 53C/70 2. [-4,3,12]/13, $[723,-500,366]/\sqrt{206685}$ $[42,60,-1]/\sqrt{5365}$ 3. [-4,3,12]/13, $[-3,-40,9]/13\sqrt{10}$, $[3,0,1]/\sqrt{10}$ 4. [3,-2,6]/7, $[-186,75,118]/7\sqrt{1105}$, $[-14,-30,-3]/\sqrt{1105}$ 5. same as 2. 6. A' = [14,30,3]/70, B' = [3,0,1]/15 C' = [42,60,-1]/-210 7. [1,-2,2]/3, $[10,-2,-7]/\sqrt{153}$, $[2,3,2]/\sqrt{17}$ 8. [2,-1,2]/3, [2,1,-2]/3, [1,2,2]/3 9. [1,-2,2]/3, $[22,1,-10]/3\sqrt{65}$, $[2,6,5]/\sqrt{65}$ 10. A = -2[2,-2,1]/9 - 29[2,1,-2]/9 + 26[1,2,2]/9.

EXERCISE 26 (page 110) 1. See answers to exercise 3.2. See answers to exercises 4 and 5 3. $\lambda = \pm 9$, $x^2 + y^2 - z^2 = 1$, (1,2,0) 4. $\lambda = -9, \pm 18$, $2x^2 - y^2 - z^2 = 0$, (13/9,5/9,7/9) 5. $\lambda = \pm 9, 18$, $2x^2 + y^2 - z^2 = 0$, (1,1,1) 6. $\lambda = 0,3,6$, $x^2 + 2y^2 = 1$, (0,-1,3) 7. $\lambda = 0, \pm 3$, $x^2 - y^2 = 1$, (1,1,1) 8. $\lambda = 0,-2,36$, $18x^2 - y^2 = 9z$, (1/3,2/3,2/3) 9. $\lambda = 0,12,-18$, $2x^2 - 3y^2 + 2z = 0$, (1,1,1) 10. $\lambda = 2,9,36$, $2x^2 + 9y^2 + 36z^2 = 36$ 11. $\lambda = -2, 9, -36$, $-2x^2 + 9y^2 - 36z^2 = 36$ 12. $\lambda = -8,9,36$, $9x^2 + 36y^2 - 8z^2 = 72$ 13. $\lambda = 0,14,24$, $7x^2 + 12y^2 = 168$ 14. $\lambda = 0,14,-24$, $7x^2 - 12y^2 = 168$ 15. $\lambda = 1,1,15$, $x^2 + y^2 + 15z^2 = 1$.

EXERCISE 27 (page 115) 2. $r^2 = 2\sec^2\alpha$, $1/T = +2\sec\alpha\csc\alpha$ 3. $[1,-1,1]/\sqrt{3}$ 4. $[-1,2]/\sqrt{5}$, $[-2,1]/\sqrt{5}$ 5. 100[-10,5,-1], 100[-10/729,5/81,-1/9] 6. $[1,2,3]/\sqrt{14}$, $[-11,-8,9]/\sqrt{266}$, $[3,-3,1]/\sqrt{19}$ 7. [0,-3/5,4/5], [1,0,0], [0,4/5,3/5] 8. 3x - y - 2z = 4 9. -z + 1 = 0, [2,0,1] + t[1,0,0].

EXERCISE 28 (page 116) 1. [-y,-x,1] 2. 5x + 2y - z = 10, [2,5,10] + t[5,2,-1] 3. x - 2y - z + 6 = 0 4. $\cos^{-1} 3/4$ 5. $x_1x/a^2 + y_1y/b^2 + z_1z/c^2 = 1$.

EXERCISE 29 (page 118) 2. (a) $-\sqrt{22}$ (b) 1/2 (c) $-4/\sqrt{5}$ (d) $4\sqrt{2}$ 3. (a) 2x + 2y + z = 9, x-2/2 = y-2/2 = z-1/1 (b) z = 1; x = 0, y = 0 (c) 2x -3z + 1 = 0; x-1/2 = y-1/0 = z-1/-3 (d) 2x + 2y - z - 2 = 0, x-1/2 = y-1/2 = z-2/-1 (e) 2x + 2y + z - 3 = 0, 3x-2/6 = 3y-2/6 = 3z-1/3 4. y + 2z = 0.

EXERCISE 30 (page 121) 3. 0,0 4. 3,0

EXERCISE 31 (page 123) 1. (a) 35/6 (b) 57/10 2. (a) $\pi/2$ (b) $\pi/2$ 3. (a) 0 (b) 1/3 4. 3π 5. -5/3.

EXERCISE 32 (page 127) 1. πab 2. 1/2 3. -1/28 4. 0 5. 13/3 6. $3\pi/2$ 7. $1/\sqrt{2}$ 8. -2 9. $3\pi a^2/8$ 10. $\sqrt{2}\pi^3/3$.

EXERCISE 33 (page 128) 1. $13\pi/3$ 2. 9π 3. $16(\pi/2 - 1)$ 4. $25\pi/2$.

EXERCISE 34 (page 132) 1. (a) $\dfrac{v^3 - 3xu^2v^2 + 4}{6x^2uv^2 + 2y}$ (b) $\dfrac{2xu^2 + 3y^3}{3x^2uv^2 + y}$ 2. $\cos\theta$, $-\sin\theta/r$ 3. 2r,0 5. $\dfrac{v - x^2}{u^2 - uv}$ 6. 0, $\pi/12$ 7. (a) $4(u^2 + v^2)$ (b) $\dfrac{u}{2(u^2 + v^2)}$ 8. $r\,dr\,d\theta\,dz$ 9. yes, $g = \ln f$.

EXERCISE 35 (page 137) 1. common value 27 2. 81π 3. 9π 4. 4π 5. 24π 6. $-\pi$ 7. 0 8. 0 9. and 10. use identities on page 118.

APPENDIX ONE

Example: Condider the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$

Now $A' = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

And $A'A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$ $\quad (A'A)^{-1} = \dfrac{\begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix}}{11}$

Then $(A'A)^{-1}A' = \dfrac{\begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix}}{11} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \dfrac{\begin{bmatrix} 4 & -7 & 1 \\ 3 & 3 & -2 \end{bmatrix}}{11}$

To verify we see that $\dfrac{\begin{bmatrix} 4 & -7 & 1 \\ 3 & 3 & -2 \end{bmatrix}}{11} \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

To determine a right inverse, we consider:

$$A'(AA')^{-1} = A'(A')^{-1}A^{-1} = A^{-1}$$

$AA' = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$ but $\Delta = 0$ here and thus no right inverse exists in this case!